Degeneration scheme of 4-dimensional Painlevé-type equations

Hiroshi Kawakami, Akane Nakamura, and Hidetaka Sakai
Graduate School of Mathematical Sciences,
University of Tokyo,
Komaba, Tokyo 153-8914, Japan.

Abstract

Four 4-dimensional Painlevé-type equations are obtained by isomonodromic deformation of Fuchsian equations: they are the Garnier system in two variables, the Fuji-Suzuki system, the Sasano system, and the sixth matrix Painlevé system [30]. Degenerating these four source equations, we systematically obtained other 4-dimensional Painlevé-type equations. If we only consider Painlevé-type equations whose associated linear equations are of unramified type, there are 22 types of 4-dimensional Painlevé-type equations: 9 of them are partial differential equations, 13 of them are ordinary differential equations. Some well-known equations such as Noumi-Yamada systems are included in this list. They are written as Hamiltonian systems, and their Hamiltonians are neatly written using Hamiltonians of the classical Painlevé equations.

Keywords. integrable system, Painlevé equations.

2000 Mathematical Subject Classification Numbers. 33E17, 34M55, 39A12.

1 Introduction

Succeeding to elliptic functions, Painlevé transcendents are studied from various viewpoints, as special functions defined by nonlinear differential equations. Usually, Painlevé equations are classified into six types. However, if we consider rational surfaces called spaces of initial values, Painlevé equations of the third type fall into three types, distinguished by number of parameters. Thus, it is reasonable to consider that there are 8 types of Painlevé equations [29].

The Painlevé equations are written in Hamiltonian systems [26]. Let us see the explicit forms of them:

$$
\begin{aligned}
t(t-1)H_{\mathrm{VI}}\left(\begin{matrix}\alpha,\beta\\ \gamma,\delta\end{matrix};t;q,p\right) =&\ q(q-1)(q-t)p^2\\
&+\{\delta q(q-1)-(2\alpha+\beta+\gamma+\delta)q(q-t)+\gamma(q-1)(q-t)\}p\\
&+\alpha(\alpha+\beta)(q-t),\\
tH_{\mathrm{V}}\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};t;q,p\right) =&\ p(p+t)q(q-1)+\beta pq+\gamma p-(\alpha+\gamma)tq,
\end{aligned}
$$

$$
H_{\mathrm{IV}}(\alpha,\beta;t;q,p) = pq(p-q-t)+\beta p+\alpha q, \quad tH_{\mathrm{III}}(D_6)(\alpha,\beta;t;q,p) = p^2q^2-(q^2-\beta q-t)p-\alpha q,
$$

$$
tH_{\mathrm{III}}(D_7)(\alpha;t;q,p) = p^2q^2+\alpha qp+tp+q, \quad tH_{\mathrm{III}}(D_8)(t;q,p) = p^2q^2+qp-q-\frac{t}{q},
$$

$$
H_{\mathrm{II}}(\alpha;t;q,p) = p^2-(q^2+t)p-\alpha q, \qquad H_{\mathrm{I}}(t;q,p) = p^2-q^3-tq.
$$

These are non-autonomous Hamiltonian system with t as independent variable, p and q as canonical variables. Corresponding autonomous systems of them can be solved by using elliptic functions.

***Remark* 1.1.** As for Painlevé equation of the fifth type, it is sometimes convenient to use another Hamiltonian for calculations:

$$t\tilde{H}_{\mathrm{V}}\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};t;q,p\right)=q(q-1)^2p^2+\{(1-q)(\alpha+(\beta+2\gamma)q)+tq\}p-\gamma(\beta+\gamma)(1-q). \tag{1.1}$$

The following canonical transformation

$$q\to 1-\frac{1}{q},\quad p\to q(pq-\gamma)$$

changes the above $\tilde{H}_{\mathrm{V}}$ into a biquadratic polynomial

$$H_{\mathrm{V}}\left(\begin{matrix}\beta+\gamma,\alpha+\beta\\ -\beta\end{matrix};t;q,p\right)-\frac{\alpha\gamma}{t}+\gamma. \qquad \square$$

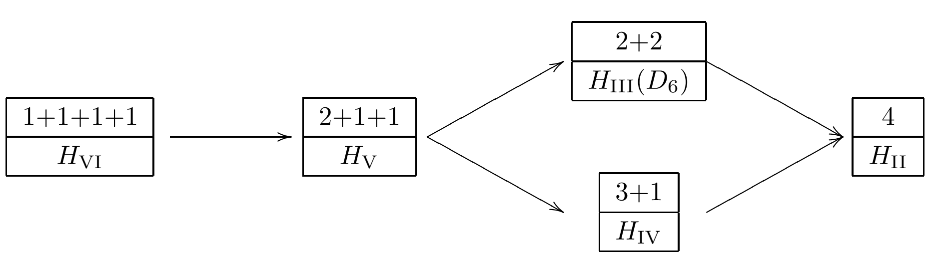
In general, nonlinear differential equation may have singular points whose positions are not determined only by the equation. In this case, positions of singular points also depend on initial values. We call such singular points movable singularities. As in the case of differential equations satisfied by elliptic functions, when movable singularities of an equation are at most poles, we say that the equation has Painlevé property. The Painlevé equations enjoy the Painlevé property. If we limit our attention to second order algebraic differential equations in the normal form and eliminate the cases when it is integrable by elementary functions or solved by solutions of linear equations or elliptic functions, there are no other equations than Painlevé equation having the Painlevé property [28, 7].

The theory of Painlevé equations is generalized to higher order nonlinear differential equations, and some important equations are proposed and investigated. For examples, Gordoa, Joshi, and Pickering proposed a higher order generalization of the second and the fourth Painlevé equations [9]. According to Koike's work, they turned out to be restrictions of independent variables of the classically known Garnier systems [21]. However, some equations such as Noumi-Yamada systems or the Sasano system still remain beyond understanding by the framework of classical Painlevé equations or Garnier systems.

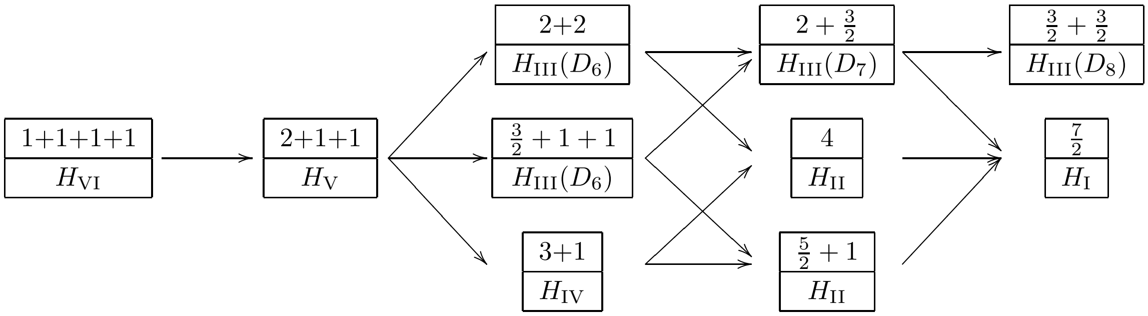
In appearance, these equations seem to be sporadic, and not well-organized compared to the case of 2-dimensional phase space. This article aims to make a theory of clear classification when the phase spaces are of four dimension.

Let us review 2-dimensional case. There are at least two ways effective for the classification. One of them is the theory of initial value spaces initiated by Okamoto. Thanks to this, we can characterize the Painlevé equations by rational sufaces [25]. These surfaces are similar to rational elliptic surfaces but are slightly different. It corresponds to the fact that autonomous limits of Painlevé equations lead to differential equations satisfied by elliptic functions. For the higher dimensional cases, more difficult theory of algebraic varieties will be needed; it might be interesting though.

We want to apply the other theory for higher-dimensional cases. That is to say, we try to approach via the deformation theory of linear differential equations. This theory was initiated by R.Fuchs, who obtained the sixth Painlevé equation from deformation of a second-order Fuchsian type equation with four singular points [4]. Non-Fuchsian equations are derived from Fuchsian equations by degeneration such as confluence of singular points. These limit procedures induce degenerations of the Painlevé equations [8, 13].



In the scheme above, we only considered confluence processes. If we also consider degenerations of Jordan canonical forms of principal parts of coefficient matrices, the degeneration scheme becomes as follows:



This scheme was obtained by Ohyama and Okumura [24]. Newly added parts require Puiseux series for formal solutions of corresponding linear equations. We say that such equations are of ramified type, and distinguish from the unramified case.

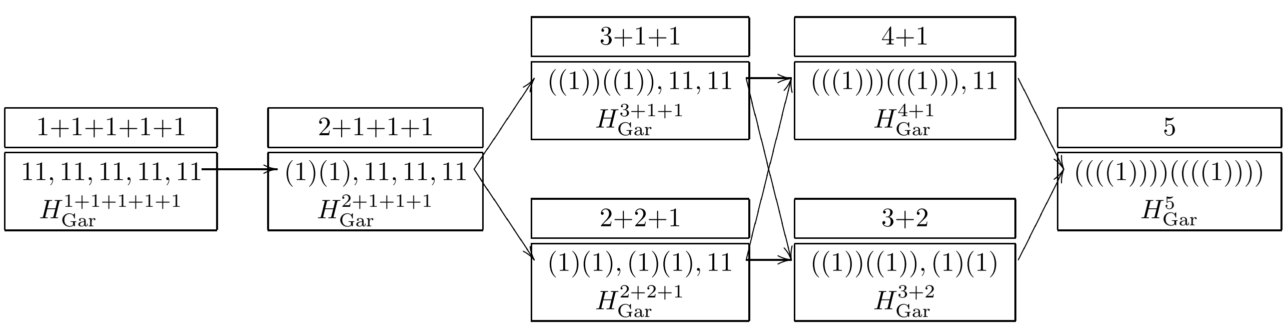
From now on, we use the term "Painlevé-type equations" for nonlinear equations derived by generalized isomonodromic deformations of linear differential equations. There are two problems when we want to use corresponding linear equations for the classification of Painlevé-type equations. One of the problems is the fact that there are more than one corresponding linear equations for one Painlevé-type equation. The other problem is the fact that the number of linear equations that we should consider is not finite, if we do not limit the sizes of linear equations.

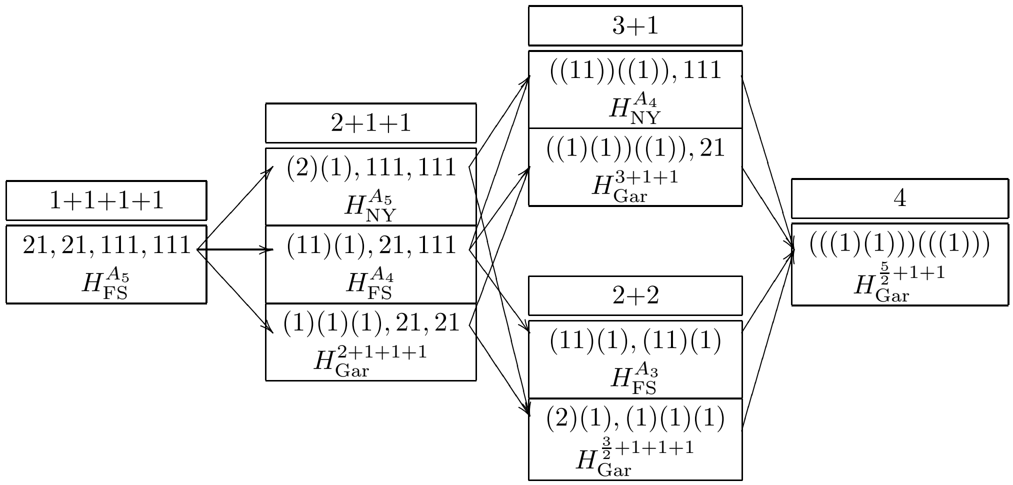
However, transformation theory of linear equations has developed, and now we can conquer these problems. Katz's two transformations called middle convolution and addition are important, and they preserve corresponding Painlevé-type equations [16, 3, 10]. If we fix the number of parameters corresponding to the dimension of phase spaces of Painlevé-type equations, Fuchsian equations fall into finite types of equations by the two transformations. In particular, when phase space is 4-dimensional, equation is equivalent to one of the 13-types via Katz's transformations. There are four non-trivial Painlevé-type equations corresponding to these linear equations. These deformation equations are expressed in forms of Hamiltonians [30].

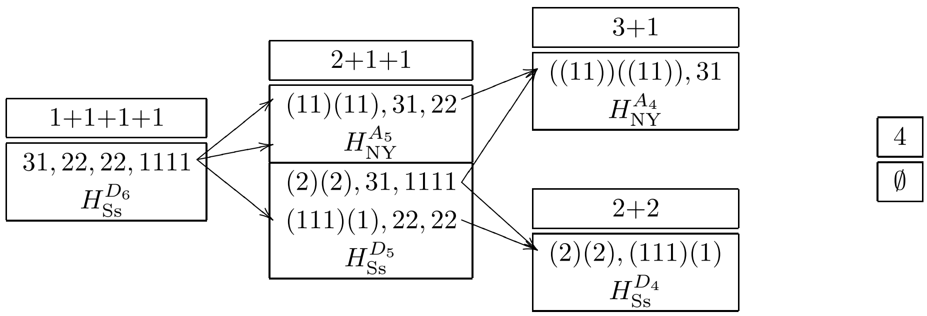
There are also several interesting works attempting to construct a similar theory for non-Fuchsian equations [1, 2, 11, 17, 36]. However, we do not take this way; we apply classification of Fuchsian equations and consider degeneration to classify Painlevé-type equations.

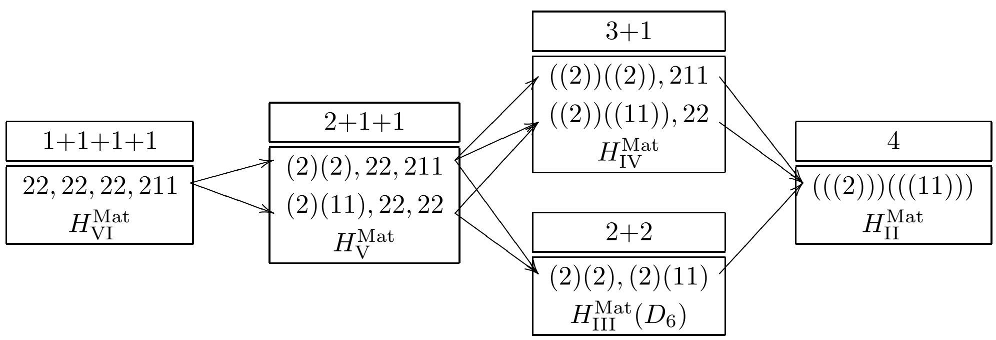
As a result, we obtained 22 types of equations, as we shall see in the next section. Among these 22 types, 9 of them are partial differential systems, and 13 are ordinary differential systems. It sometimes happen that different degenerations yield the same Painlevé-type equation. In these cases, corresponding linear equations transform to one another by the Laplace transformation. We shall see such interesting topics in the last section.

Here, let us see the degeneration scheme beforhand.









The symbols used in the scheme will be explained in Section 3. The theory of isomonodromic deformation has been well studied since the work of Jimbo, Miwa, and Ueno [15]. In their article, eigenvalues of leading terms of linear equations are assumed to be distinct. To classify Painlevé-type equations, however, we introduce a notion of spectral type, and also consider cases when such eigenvalues are not necessarily distinct. If we consider examples such as generalized hypergeometric function, it is natural to include these cases. In this article, we express Hamiltonians of Painlevé-type equations by Hamiltonians of classical Painlevé equations. These are very effective tools to identify equations.

This article is organized as follows. In the next section, we introduce 22 types of Hamiltonian systems. In the third section, we explain local data of linear equations. In the fourth section, we explain procedure of degeneration through confluences of singularities. In the fifth section, we show Lax pairs for Painlevé-type equations. In the last section, we comment on several things that we should pay attentions to. Detailed calculations of degenerations are written in the appendix.

We only deal with equations of unramified-type. More equations are derived from ramified linear equations [19].

Acknowledgments

The author would like to thank Prof. Oshima, Prof. Okamoto, Prof. Ohyama, Dr. Hiroe, Prof. Suzuki, Prof. Tsuda, Prof. Jimbo, Prof. Nishioka, Prof. Takemura, Prof. Yamakawa, for valuable discussion and advice. This work was partially supported by Grant-in-Aid no.24540205, the Japan Society for the Promotion of Science.

2 List of Hamiltonians

There are 22 Hamiltonian systems that we treat in this paper. However these do not include the case that associated linear equations have singularities of ramified type. We will deal with only the unramified cases in this paper. We expect a further research including ramified case. Before we go into the explanation about the theory, let us see the expressions of the 22 Hamiltonians.

In the first place we look at the Garnier system and degenerate Garnier systems, which are classical systems found in the early 20th century. On the degeneration scheme of this family, a detailed study is now well known [20], though we introduce new expressions for some of these systems.

The first system is the Garnier system, which is obtained from a deformation of a Fuchsian equation with 5 regular singular points [8]. In the original paper of R. Garnier, the dependent variables are the positions of apparent singular points. However the equation does not have Painlevé property, that is, the solution has movable algebraic singularities. H. Kimura and K. Okamoto used symmetric functions of apparent singularities as dependent variables, so that the Hamiltonian system enjoys the Painlevé property [22]. The following Hamiltonian coincides with theirs, although they do not use the Hamiltonian of the sixth Painlevé equation:

$$
\begin{aligned}
(2.1)\qquad & t_i(t_i-1)H^{1+1+1+1+1}_{Gar,t_i}\begin{pmatrix}\alpha,\beta & t_1 & q_1,p_1\\ \gamma_1,\gamma_2,\delta\,; & t_2\,; & q_2,p_2\end{pmatrix}\\
&= t_i(t_i-1)H_{\mathrm{VI}}\begin{pmatrix}\alpha,\beta & \\ \gamma_i,\gamma_{i+1}+\delta & ;t_i;q_i,p_i\end{pmatrix} + (2q_ip_i+q_{i+1}p_{i+1}+\alpha+\gamma_1+\gamma_2+\delta-1)q_1q_2p_{i+1}\\
&\quad -\frac{q_1q_2}{t_i-t_{i+1}}(t_i(t_i-1)p_i^2+2t_i(t_{i+1}-1)p_1p_2+t_{i+1}(t_i-1)p_{i+1}^2)\\
&\quad +\gamma_{i+1}\frac{t_{i+1}(t_i-1)}{t_i-t_{i+1}}q_i(p_i-p_{i+1})-\frac{\gamma_it_i}{t_i-t_{i+1}}((t_1-1)p_1+(t_2-1)p_2), \qquad (i\in\mathbb{Z}/2\mathbb{Z}).
\end{aligned}
$$

Relation between canonical variables and parameters of associated linear equations will be explained later.

The second is the degenerate Garnier system obtained by a confluence of two regular singular points;

$$
\begin{aligned}
(2.2)\qquad H^{2+1+1+1}_{Gar,\,t_1}\begin{pmatrix}\alpha,\beta & t_1 & q_1,p_1\\ \gamma,\delta\,; & t_2\,; & q_2,p_2\end{pmatrix} &= H_{\mathrm{V}}\begin{pmatrix}-\alpha-\gamma,-\beta-\gamma-\delta-1 & \\ \alpha+\beta+\gamma & ;t_1;q_1,p_1\end{pmatrix}\\
&\quad+\frac{p_1}{t_1}[\gamma(q_1-q_2)+p_2q_2(q_2-1)]\\
&\quad+\frac{1}{t_1-t_2}\left((q_1-q_2)p_1-\beta\right)\left((q_2-q_1)p_2-\gamma\right),\\
(2.3)\qquad H^{2+1+1+1}_{Gar,\,t_2}\begin{pmatrix}\alpha,\beta & t_1 & q_1,p_1\\ \gamma,\delta\,; & t_2\,; & q_2,p_2\end{pmatrix} &= H_{\mathrm{V}}\begin{pmatrix}-\alpha-\beta,-\beta-\gamma-\delta-1 & \\ \alpha+\beta+\gamma & ;t_2;q_2,p_2\end{pmatrix}\\
&\quad+\frac{p_2}{t_2}[\beta(q_2-q_1)+p_1q_1(q_1-1)]\\
&\quad+\frac{1}{t_2-t_1}\left((q_1-q_2)p_1-\beta\right)\left((q_2-q_1)p_2-\gamma\right).
\end{aligned}
$$

***Remark* 2.1.** These canonical variables are different form the hitherto known [20]. The original Hamilto-

nians were written as

$$
\begin{aligned}
s_1^2 H_1 = {} & \lambda_1^2(\lambda_1 - s_1)\mu_1^2 + 2\lambda_1^2\lambda_2\mu_1\mu_2 + \lambda_1\lambda_2(\lambda_2 - s_2)\mu_2^2 \\
& - \{(\theta_0^\infty + \theta_2 - 1)\lambda_1^2 + \theta_1^\infty\lambda_1(\lambda_1 - s_1) + \eta(\lambda_1 - s_1) + \eta s_1\lambda_2\}\mu_1 \\
& - \{(\theta_0^\infty + \theta_1^\infty - 1)\lambda_1\lambda_2 + \theta_2\lambda_1(\lambda_2 - s_2) - \eta(s_2 - 1)\lambda_2\}\mu_2 + \theta^\infty\lambda_1, \\
s_2(s_2 - 1)H_2 = {} & \lambda_1^2\lambda_2\mu_1^2 + 2\lambda_1\lambda_2(\lambda_2 - s_2)\mu_1\mu_2 \\
& + \left\{\lambda_2(\lambda_2 - 1)(\lambda_2 - s_2) + \frac{s_2(s_2 - 1)}{s_1}\lambda_1\lambda_2\right\}\mu_2^2 \\
& - \{(\theta_0^\infty + \theta_1^\infty - 1)\lambda_1\lambda_2 + \theta_2\lambda_1(\lambda_2 - s_2) - \eta(s_2 - 1)\lambda_2\}\mu_1 \\
& - \left\{(\theta_0^\infty - 1)\lambda_2(\lambda_2 - 1) + \theta_1^\infty\lambda_2(\lambda_2 - s_2) + \theta_2(\lambda_2 - 1)(\lambda_2 - s_2)\right. \\
& \left. + \frac{s_2(s_2 - 1)}{s_1}(\theta_2\lambda_1 + \eta\lambda_2)\right\}\lambda_2 + \theta^\infty\lambda_2.
\end{aligned}
$$

This expression is not symmetric in the canonical variables, while the Hamiltonians (2.2)–(2.3) are symmetric. Besides, they are expressed simply by using the Hamiltonian of the fifth Painlevé equation. The other degenerate Garnier systems below are expressed in a similar fashion. $\square$

The next Hamiltonians are ones of the degenerate Garnier system associated to a linear equation with two irregular singular points and one regular singular point:

$$
\begin{aligned}
& t_1 H^{2+2+1}_{Gar,t_1}\begin{pmatrix} \alpha,\beta & t_1 & q_1,p_1 \\ \gamma & t_2 & q_2,p_2 \end{pmatrix} \\
& = t_1 H_{\mathrm{V}}\begin{pmatrix} \alpha+\beta+\gamma,\beta-\alpha \\ -\beta-\gamma \end{pmatrix}; t_1; q_1, p_1 \Big) \\
& \quad + q_1q_2(p_1q_1 - \alpha) + p_2q_2(\alpha + p_1 - 2p_1q_1) - \frac{t_2}{t_1}p_1(p_2 - q_1),
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
& t_2 H^{2+2+1}_{Gar,t_2}\begin{pmatrix} \alpha,\beta & t_1 & q_1,p_1 \\ \gamma & t_2 & q_2,p_2 \end{pmatrix} \\
& = t_2 H_{\mathrm{III}}(D_6)\left(-\alpha-\beta-\gamma, -\beta; t_2; q_2, p_2\right) - (p_1q_1 - \alpha)q_2(q_1 - 1) + \frac{t_2}{t_1}p_1(p_2 - q_1).
\end{aligned} \tag{2.5}
$$

The following is associated to a linear equation with two regular singular points and one irregular singular point:

$$
\begin{aligned}
& H^{3+1+1}_{Gar,t_1}\begin{pmatrix} \alpha,\beta & t_1 & q_1,p_1 \\ \gamma & t_2 & q_2,p_2 \end{pmatrix} \\
& = H_{\mathrm{IV}}\left(\alpha, \gamma; t_1; q_1, p_1\right) + p_2q_2p_1 + \frac{1}{t_1 - t_2}\{p_1(q_1 - q_2) - \alpha\}\{p_2(q_2 - q_1) - \beta\},
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
& H^{3+1+1}_{Gar,t_2}\begin{pmatrix} \alpha,\beta & t_1 & q_1,p_1 \\ \gamma & t_2 & q_2,p_2 \end{pmatrix} \\
& = H_{\mathrm{IV}}\left(\beta, \gamma; t_2; q_2, p_2\right) + p_1q_1p_2 + \frac{1}{t_2 - t_1}\{p_1(q_1 - q_2) - \alpha\}\{p_2(q_2 - q_1) - \beta\}.
\end{aligned} \tag{2.7}
$$

The following system is the degenerate Garnier system associated to a linear equation with only two irregular singular points:

$$
H^{3+2}_{Gar,t_1}\left(\alpha,\beta; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} q_1,p_1 \\ q_2,p_2 \end{matrix}\right) = H_{\mathrm{III}}(D_6)\left(-\beta, \alpha+1; t_1; q_1, p_1\right) - p_1 - \frac{q_1q_2}{t_1}(q_2 - p_2 + t_2) + p_1p_2 - q_2, \tag{2.8}
$$

$$
H^{3+2}_{Gar,t_2}\left(\alpha,\beta; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} q_1,p_1 \\ q_2,p_2 \end{matrix}\right) = H_{\mathrm{IV}}\left(\alpha, \beta; t_2; q_2, p_2\right) - p_1q_1(p_2 - 2q_2 - t_2) - q_1q_2 + t_1p_1. \tag{2.9}
$$

The next system is associated to a linear equation with one regular singular point and one irregular point:

$$H^{4+1}_{Gar,t_1}\left(\alpha,\beta;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)=H_{\mathrm{II}}\left(-\beta;t_1;q_1,p_1\right)+p_2q_2(q_1-q_2+t_2)+p_1p_2+\alpha q_2, \tag{2.10}$$

$$\begin{aligned}&H^{4+1}_{Gar,t_2}\left(\alpha,\beta;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)\\&=-p_2^2q_2-t_2p_2q_2^2+t_2^2p_2q_2+\alpha t_2q_2-\beta p_2+p_1p_2(q_1-2q_2+t_2)+q_1q_2(p_2q_2-\alpha)+\alpha p_1+t_1p_2q_2.\end{aligned} \tag{2.11}$$

Remark 2.2. In the expression above we did not use a Painlevé Hamiltonian. If we do so, we need power functions. When we change the variables as

$$t_2\to t_2^{\frac{2}{3}},\quad q_2\to t_2^{-\frac{1}{3}}p_2,\quad p_2\to -t_2^{\frac{1}{3}}q_2,$$

the second Hamiltonian becomes

$$\begin{aligned}H^{4+1}_{Gar,t_2}\left(\alpha,\beta;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)=\frac{2}{3}\big[&H_{\mathrm{IV}}\left(\beta,\alpha;t_2;q_2,p_2\right)-\frac{p_2q_2}{2t_2}-p_1q_2(q_1-2t_2^{-\frac{1}{3}}p_2+t_2^{\frac{2}{3}})\\&-t_2^{-\frac{2}{3}}q_1p_2(p_2q_2+\alpha)+\alpha t_2^{-\frac{1}{3}}p_1-t_1t_2^{-\frac{1}{3}}p_2q_2\big].\end{aligned} \tag{2.12}$$

Here appears the fourth Painlevé Hamiltonian. □

Concerning a linear equation with only one irregular singular point, which is obtained by a confluence of all singularities, we have the following system:

$$H^{5}_{Gar,t_1}\left(\alpha;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)=-q_1(q_1p_1-\alpha)+q_2(q_1(p_2+q_2)-2p_1+t_1)+p_1(p_2-2t_2), \tag{2.13}$$

$$H^{5}_{Gar,t_2}\left(\alpha;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)=H_{\mathrm{IV}}\left(-1,\alpha;2t_2;q_2,p_2\right)+q_1q_2(q_1q_2-2p_1+t_1)+p_1(p_1-p_2q_1-t_1). \tag{2.14}$$

We have 7 systems which have two independent variables so far. In Kimura's original paper [20], there are 8 systems, but the last system denoted as (9/2) is obtained from a deformation of a linear equation with singular point of ramified type; so we put aside this one.

In the degeneration scheme, further two degenerate Garnier systems appear. These two systems are associated to linear systems of rank two with singular points of ramified type but these are also associated to unramified linear systems of rank three. These rank three systems are obtained by confluence of singularities from Fuji-Suzuki system. With respect to degenerate Garnier systems of ramified type, all of them are obtained by H. Kawamuko and there are 9 systems in all [18].

The first one of them is associated to a linear equation with two irregular singular points, which is obtained from degereration of linear system of rank three associated to Fuji-Suzuki system:

$$\begin{aligned}&t_1H^{\frac{3}{2}+1+1+1}_{Gar,t_1}\left(\begin{matrix}\alpha,\beta\\\gamma\end{matrix};\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)\\&=t_1H_{\mathrm{III}}(D_6)\left(-\alpha,\gamma-\alpha;t_1;q_1,p_1\right)+q_1(q_1p_1p_2-\alpha p_2)+\frac{t_1}{t_1-t_2}(p_1(q_1-q_2)-\alpha)(p_2(q_2-q_1)-\beta),\end{aligned} \tag{2.15}$$

$$\begin{aligned}&t_2H^{\frac{3}{2}+1+1+1}_{Gar,t_2}\left(\begin{matrix}\alpha,\beta\\\gamma\end{matrix};\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)\\&=t_2H_{\mathrm{III}}(D_6)\left(-\beta,\gamma-\beta;t_2;q_2,p_2\right)+q_2(q_2p_1p_2-\beta p_1)+\frac{t_2}{t_2-t_1}(p_1(q_1-q_2)-\alpha)(p_2(q_2-q_1)-\beta).\end{aligned} \tag{2.16}$$

The second one is associated to a linear equation possessing only one irregular singular point, which is

obtained by confluence of all singular points from a linear equation associated to Fuji-Suzuki system:

$$
\begin{aligned}
&H^{\frac{5}{2}+1+1}_{Gar,t_1}\left(\alpha,\beta;\begin{matrix}t_1\\ t_2\end{matrix};\begin{matrix}q_1,p_1\\ q_2,p_2\end{matrix}\right) \\
&\quad= H_{\mathrm{II}}\left(-\alpha;t_1;q_1,p_1\right)+p_1p_2+\frac{1}{t_1-t_2}(p_1(q_1-q_2)-\alpha)(p_2(q_2-q_1)-\beta),
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
&H^{\frac{5}{2}+1+1}_{Gar,t_2}\left(\alpha,\beta;\begin{matrix}t_1\\ t_2\end{matrix};\begin{matrix}q_1,p_1\\ q_2,p_2\end{matrix}\right) \\
&\quad= H_{\mathrm{II}}\left(-\beta;t_2;q_2,p_2\right)+p_1p_2+\frac{1}{t_2-t_1}(p_1(q_1-q_2)-\alpha)(p_2(q_2-q_1)-\beta).
\end{aligned} \tag{2.18}
$$

We have already seen the 9 partial differential systems with two independent variables so far, and we will see the 13 ordinary differential systems below.

We have three more systems which are obtained from the isomonodromic deformation of Fuchsian equations, apart from the Garnier system. These are the Fuji-Suzuki system, the Sasano system, and the system which we call the sixth matrix Painlevé system. We begin with the Fuji-Suzuki system. It was obtained from the similarity reduction of a Drinfel'd-Sokolov hierarchy [5]:

$$
\begin{aligned}
&t(t-1)H^{A_5}_{FS}\left(\begin{matrix}\alpha,\beta,\gamma\\ \delta,\epsilon,\omega\end{matrix};t;\begin{matrix}q_1,p_1\\ q_2,p_2\end{matrix}\right) \\
&= t(t-1)H_{\mathrm{VI}}\left(\begin{matrix}\alpha,\beta+\delta\\ \beta+\gamma,\epsilon-\omega+1\end{matrix};t;q_1,p_1\right)+t(t-1)H_{\mathrm{VI}}\left(\begin{matrix}\beta,\alpha+\delta\\ \alpha+\gamma,\epsilon-\alpha+1\end{matrix};t;q_2,p_2\right) \\
&\quad+(q_1-t)(q_2-1)\{(p_1q_1-\alpha)p_2+p_1(p_2q_2-\beta)\}.
\end{aligned} \tag{2.19}
$$

***Remark* 2.3.** Before the result of Fuji and Suzuki, Tsuda calculated isomonodromic deformations with respect to a class of Fuchsian equations including 21,21,111,111 from a viewpoint of reduction theory of his UC hierarchy. Although his system of equations was not written in the form of the Hamiltonian system, it was found that his system includes the Fuji-Suzuki's coupled sixth Painlevé system in the paper [30]. In the aftermath Tsuda gave a Hamiltonian expression to the whole systems in his class [35]. In this paper we will call this system the Fuji-Suzuki system because they were the first ones who gave the expression of the coupled sixth Painlevé system. □

A degeneration from the above Fuji-Suzuki system produces the following Hamiltonian:

$$
\begin{aligned}
&tH^{A_4}_{FS}\left(\begin{matrix}\alpha,\beta,\gamma\\ \delta,\epsilon\end{matrix};t;\begin{matrix}q_1,p_1\\ q_2,p_2\end{matrix}\right) \\
&= tH_{\mathrm{V}}\left(\begin{matrix}\alpha+\beta+\delta+\epsilon,\alpha+\gamma-\delta-1\\ -\alpha-\epsilon\end{matrix};t;q_1,p_1\right)+tH_{\mathrm{V}}\left(\begin{matrix}\alpha+\epsilon,\alpha+\gamma-\epsilon-1\\ -\alpha\end{matrix};t;q_2,p_2\right) \\
&\quad+p_1(q_2-1)(p_2(q_1+q_2)-\epsilon).
\end{aligned} \tag{2.20}
$$

Further degeneration also produces the following Hamiltonian:

$$
\begin{aligned}
tH^{A_3}_{FS}\left(\alpha,\beta,\gamma,\delta;t;q_1,p_1,q_2,p_2\right) &= tH_{\mathrm{III}}(D_6)\left(\alpha+\gamma,-\beta+\gamma;t;q_1,p_1\right)+tH_{\mathrm{III}}(D_6)\left(\delta,-\beta+\delta;t;q_2,p_2\right) \\
&\quad+p_1q_2(p_2(q_1+q_2)+\delta).
\end{aligned} \tag{2.21}
$$

We call Hamiltonian systems of $H^{A_4}_{FS}$ and $H^{A_3}_{FS}$ degenerate Fuji-Suzuki systems. Special function solutions of these equations are calculated in the paper [33].

The famous Noumi-Yamada systems are written in the form of Hamiltonian system by using the following

Hamiltonians:

(2.22)
$$tH_{NY}^{A_5}\begin{pmatrix}\alpha,\beta,\gamma & & q_1,p_1\\ \delta,\epsilon & ;t; & q_2,p_2\end{pmatrix} = tH_{\rm V}\begin{pmatrix}\alpha+\beta,\alpha+\gamma+\epsilon & ;t;q_1,p_1\\ -\alpha & \end{pmatrix} + tH_{\rm V}\begin{pmatrix}\alpha+\gamma+\delta,\alpha+\gamma+\epsilon & ;t;q_2,p_2\\ -\alpha-\gamma & \end{pmatrix}$$
$$+ 2p_1p_2q_1(q_2-1),$$

$$H_{NY}^{A_4}\begin{pmatrix}\alpha,\beta & & q_1,p_1\\ \gamma,\delta & ;t; & q_2,p_2\end{pmatrix} = H_{\rm IV}(\beta,\alpha;t;q_1,p_1) + H_{\rm IV}(\delta,\alpha+\gamma;t;q_2,p_2) + 2q_1p_1p_2. \tag{2.23}$$

These two systems are obtained by degenerations from both the Fuji-Suzuki system and the Sasano system.

***Remark* 2.4.** The Noumi-Yamada systems are well known in the following expression:

$$NY^{A_4} \ : \ \frac{df_i}{dt} = f_i(f_{i+1}-f_{i+2}+f_{i+3}-f_{i+4})+\alpha_i, \quad (i\in\mathbb{Z}/5\mathbb{Z}), \tag{2.24}$$

$$NY^{A_5} \ : \ \frac{df_i}{dt} = f_i(f_{i+1}f_{i+2}-f_{i+2}f_{i+3}+f_{i+1}f_{i+4}-f_{i+2}f_{i+5}+f_{i+3}f_{i+4}-f_{i+4}f_{i+5}) \tag{2.25}$$
$$+(-1)^i(\alpha_{i+1}+\alpha_{i+2}+\alpha_{i+5})f_i+\alpha_i(f_i+f_{i+2}+f_{i+4}), \quad (i\in\mathbb{Z}/6\mathbb{Z}),$$

as systems of equations for the unknown function $f_0,\dots,f_l,(l=4$ or $5)$ [23]. These systems coincide with the Hamiltonians above by putting $p_1=f_2$, $q_1=-f_1$, $p_2=f_4$, and $q_2=-f_1-f_3$. Here the parameters are $\alpha=-\alpha_1$, $\beta=-\alpha_2$, $\gamma=-\alpha_3$, $\delta=-\alpha_4$, $\epsilon=-\alpha_5$ [32]. □

We introduce another coupled Painlevé system called Sasano system. Sasano systems were obtained by a generalization of space of initial values for the Painlevé equations [31], and it is also calculated from similarity reduction of Drinfel'd-Sokolov hierarchy [6]:

$$H_{Ss}^{D_6}\begin{pmatrix}\alpha,\beta,\gamma & & q_1,p_1\\ \delta,\epsilon,\zeta & ;t; & q_2,p_2\end{pmatrix} = H_{\rm VI}\begin{pmatrix}\beta+\gamma+2\delta+\epsilon+\zeta,-\beta-\zeta & ;t;q_1,p_1\\ -\beta-2\gamma-2\delta-\epsilon,1-\alpha-\beta-2\delta-\epsilon-\zeta & \end{pmatrix} \tag{2.26}$$
$$+H_{\rm VI}\begin{pmatrix}\gamma+\delta,\epsilon & ;t;q_2,p_2\\ \zeta,1-\alpha-\gamma & \end{pmatrix}$$
$$+\frac{2}{t(t-1)}(q_1-1)p_2q_2\{(q_1-t)p_1-\beta-\gamma-2\delta-\epsilon-\zeta)\}.$$

This system is different from the Fuji-Suzuki system of type $A_5^{(1)}$ in its "coupling term".

Degenerations give the following Hamiltonians:

$$tH_{Ss}^{D_5}\begin{pmatrix}\alpha,\beta,\gamma, & & q_1,p_1\\ \delta,\epsilon & ;t; & q_2,p_2\end{pmatrix} = tH_{\rm V}\begin{pmatrix}\epsilon,\alpha-\beta & ;t;q_1,p_1\\ \beta & \end{pmatrix} \tag{2.27}$$
$$+tH_{\rm V}\begin{pmatrix}-2\alpha-3\beta-\gamma-\delta-2\epsilon,-\alpha-\beta-2\delta & ;t;q_2,p_2\\ \alpha+2\beta+\delta+\epsilon & \end{pmatrix}$$
$$+2p_2q_1(p_1(q_1-1)-\beta-\epsilon),$$

$$tH_{Ss}^{D_4}\begin{pmatrix}\alpha,\beta, & & q_1,p_1\\ \gamma,\delta, & ;t; & q_2,p_2\end{pmatrix} = tH_{\rm III}(D_6)\,(\alpha+\beta+\gamma,-\alpha-2\delta;t;q_1,p_1) \tag{2.28}$$
$$+tH_{\rm III}(D_6)\,(-\gamma,-\alpha-2\gamma;t;q_2,p_2)+2p_2q_1(p_1q_1+\alpha+\beta+\gamma).$$

In this article, we call these two systems degenerate Sasano systems.

At the end we introduce five Hamiltonian systems which we call matrix Painlevé systems:

$$t(t-1)H_{\rm VI}^{Mat}\begin{pmatrix}\alpha,\beta,\gamma & & q_1,p_1\\ \delta,\omega & ;t; & q_2,p_2\end{pmatrix} = {\rm tr}\left[Q(Q-1)(Q-t)P^2+\{(\delta-(\alpha-\omega)K)Q(Q-1)\right. \tag{2.29}$$
$$\left.+\gamma(Q-1)(Q-t)-(2\alpha+\beta+\gamma+\delta)Q(Q-t)\}P+\alpha(\alpha+\beta)Q\right],$$

$$tH_{\mathrm{V}}^{Mat}\begin{pmatrix}\alpha,\beta & ;t; & q_1,p_1\\ \gamma,\omega & & q_2,p_2\end{pmatrix}=\mathrm{tr}[Q(Q-1)P(P+t)+\beta QP+\gamma P-(\alpha+\gamma)tQ], \tag{2.30}$$

$$H_{\mathrm{IV}}^{Mat}\left(\alpha,\beta,\omega;t;\begin{matrix}q_1,p_1\\ q_2,p_2\end{matrix}\right)=\mathrm{tr}[QP(P-Q-t)+\beta P+\alpha Q], \tag{2.31}$$

$$tH_{\mathrm{III}(D_6)}^{Mat}\left(\alpha,\beta,\omega;t;\begin{matrix}q_1,p_1\\ q_2,p_2\end{matrix}\right)=\mathrm{tr}[Q^2P^2-(Q^2+\beta Q-t)P-\alpha Q], \tag{2.32}$$

$$H_{\mathrm{II}}^{Mat}\left(\alpha,\omega;t;\begin{matrix}q_1,p_1\\ q_2,p_2\end{matrix}\right)=\mathrm{tr}[P^2-(Q^2+t)P-\alpha Q]. \tag{2.33}$$

Here the matrices P and Q satisfy the relation $[P,Q]=(\alpha-\omega)K \quad (K=\mathrm{diag}(1,-1))$, and canonical variables can be written as

$$p_1=\mathrm{tr}P,\quad q_1=\frac{1}{2}\mathrm{tr}Q,\quad p_2=\frac{P_{12}}{Q_{12}},\quad q_2=-Q_{12}Q_{21}.$$

We denote $(1,2)$-component of matrix P as P_{12}, and so on.

3 Local data of linear equations

In this section, we present the notion of spectral type which is needed to classify linear differential equations. We consider the following first-order system:

$$\frac{dY}{dx}=A(x)Y, \tag{3.1}$$

where $A(x)$ is an $m\times m$ matrix whose entries are rational functions in x.

In the case of Fuchsian systems, in addition to the size of matrix and the number of singular points, the multiplicity of characteristic exponents at each singularity plays an important role. Let us consider the system of Schlesinger normal form, namely, the system of linear differential equations whose coefficient $A(x)$ has the following form

$$A(x)=\sum_{i=1}^{n}\frac{A_i}{x-u_i}. \tag{3.2}$$

Here A_i is an $m\times m$ matrix. In this case, the spectral type is an $(n+1)$-tuple of partitions of m:

$$m_1^1m_2^1\dots m_{l_1}^1, m_1^2\dots m_{l_2}^2,\dots,m_1^n\dots m_{l_n}^n, m_1^\infty\dots m_{l_\infty}^\infty,\qquad \sum_{j=1}^{l_i}m_j^i=m\ \text{ for }\ i=1,\dots,n,\infty.$$

This means there are m_j^i $(1\le j\le l_i)$ identical characteristic exponents at singular point $x=u_i$. Here we have assumed that A_i's and $A_\infty:=-\sum_{i=1}^n A_i$ are diagonalizable.

More generally, we would like to define the spectral type for a system of linear differential equations with irregular singularities, i.e., $A(x)$ is a matrix of general rational function in x. In such a case, what corresponds to each singular point is not merely a partition, but a *refining sequence of partitons*. Let $\lambda=(\lambda_1,\dots,\lambda_p)$ and $\mu=(\mu_1,\dots,\mu_q)$ be partitions of a natural number m, namely, $\lambda_1+\dots+\lambda_p=\mu_1+\dots+\mu_q=m$. Here λ_i's and μ_i's are not necessarily put in descending or ascending order. If the index set of λ is divided into disjoint union $\{1,2,\dots,p\}=I_1\coprod\cdots\coprod I_q$ and $\mu_k=\sum_{j\in I_k}\lambda_j$ holds, then we call λ a refinement of μ. Let $[p_0,\dots,p_r]$ be an $(r+1)$-tuple of partions of m. When p_{i+1} is refinement of p_i for all i $(i=0,\dots,r-1)$, we call $[p_0,\dots,p_r]$ a refining sequence of partions.

Now we introduce a convenient notation to express a refining sequence of partitions. Let us take a sequence

$$[(6,4,2),(4,2,4,1,1),(2,2,2,3,1,1,1)]$$

as an example. First, write the rightmost partition:

$$2223111.$$

Second, put the numbers that are grouped together in the central partition in parentheses:

$$(22)(2)(31)(1)(1).$$

Finally, put the numbers that are grouped together in the leftmost partition in parentheses:

$$((22)(2))((31))((1)(1)).$$

Thus we can express the sequence as $((22)(2))((31))((1)(1))$.

In what follows we briefly outline how the formal canonical form at an irregular singularity is computed, and how the canonical form is described by the refining sequence of partitions.

Now suppose that the equation (3.1) has a singularity at the origin. Then we have the Laurent expansion

$$\frac{dY}{dx} = \left(\frac{A_0}{x^{r+1}} + \frac{A_1}{x^r} + \cdots \right) Y, \tag{3.3}$$

where A_j $(j = 0, 1, \ldots)$ is an $m \times m$ matrix. Here we assume A_0 is diagonalizable (see the next Remark). Thus we choose the gauge of (3.3) so that A_0 is diagonal. We denote eigenvalues of A_0 by $t_1^0, \ldots, t_m^0$. Since we are interested in irregular singular case, we assume $r > 0$. It is known that if

$$t_i^0 \neq t_j^0 \quad (1 \leq i \leq l,\ l+1 \leq j \leq m),$$

then the equation (3.3) can be converted into

$$\frac{dZ}{dx} = \left(\frac{B_0}{x^{r+1}} + \frac{B_1}{x^r} + \cdots \right) Z,$$

where

$$B_i = \begin{pmatrix} B_{11}^i & O \\ O & B_{22}^i \end{pmatrix},\ B_{11}^i \in M_l(\mathbb{C}),\ B_{22}^i \in M_{m-l}(\mathbb{C})$$

under a gauge transformation by a formal power series

$$Y = P(x)Z, \quad P(x) = I + P_1 x + P_2 x^2 + \cdots.$$

Applying the procedure repeatedly, we can formally decompose the equation (3.3) into the direct sum of systems in each of which the leading term has only one distinct eigenvalue (block diagonalization). If the leading term of some block is diagonalizable (i.e., scalar matrix), we can cancel out the term by a scalar gauge transformation, thus the block reduces to the one with a lower value of r.

***Remark* 3.1.** When the leading term of (3.3) is not diagonalizable, in order to make a formal solution, we generally need to take an appropriate covering $x = \xi^k$ and to take a shearing transformation with respect to ξ. Then $P(x)$ becomes a Puiseux series. We call the linear equation with such a singularty ramified type. A linear equation that does not need a Puiseux series is called unramified type. As we have mentioned in section 1, we consider only the linear equation of unramified type in the present paper. □

Under the assumption of unramifiedness, the equation (3.3) can be transformed into

$$\frac{dY}{dx} = \left(\frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x} + \cdots \right) Y.$$

Here T_j's are diagonal matrices and $T_0 = A_0$. Since this equation is a direct sum of single equations, we can cancel terms other than the principal part by means of suitable gauge transformation by a diagonal matrix whose entries are formal power series. As a result, it turned out that equation (3.3) can be transformed into the following canonical form

$$\frac{dY}{dx} = \left(\frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x}\right) Y \tag{3.4}$$

under a gauge transformation by some formal power series.

If we write the diagonal entries of T_i as t_j^i $(j = 1, \ldots, m)$, we can express the canonical form as follows:

$$\begin{array}{cccc} \multicolumn{4}{c}{\overbrace{\hphantom{t_1^0 \quad t_1^1 \quad \ldots \quad t_1^r}}^{x=0}} \\ t_1^0 & t_1^1 & \ldots & t_1^r \\ \vdots & \vdots & & \vdots \\ t_m^0 & t_m^1 & \ldots & t_m^r \end{array}.$$

The table which is made by arranging all canonical forms at each singular point is called Riemann scheme. As we can see from the procedure to obtain the canonical form, the canonical form has a nested structure; the leftmost column is divided into some groups, and in the next column, the corresponding groups are divided into some groups again, and so on. The refining sequence of partitions of m that express the multiplicity of t_j^i is called spectral type at singularity $x = 0$. A tuple of spectral types of all singular points (separated by ",") is called a spectral type of the equation.

The following matrix

$$\exp\left(-\frac{T_0}{rx^r} + \cdots - \frac{T_{r-1}}{x}\right) x^{T_r}$$

is a fundamental solution matrix of (3.4). The degree of the polynomial in exponential function with respect to x^{-1} is called the Poincaré rank at the singular point $x = 0$. The Poincaré rank of a regular singular point is 0. In ramified case, this part becomes a polynomial in $x^{-1/k}$, then the Poincaré rank is non-integer rational number. When we want to express only the Poincaré rank of each singular point, we attach the number "Poincaré rank plus 1" to each singular point and connect them with "+". We call it the singularity pattern of the equaion. At the unramified singularity, the number "Poincaré rank plus 1" is equal to the number of partitions in the refining sequence of partitions. In this article, we use singularity patterns and spectral types to specify linear equations.

For example, in the degeneration scheme, $H_{NY}^{A_4}$ is written with the singularity pattern $3+1$ and the spectral type $((11))((1)), 111$. It means, the associated linear equation of $H_{NY}^{A_4}$ has one irregular singularity of Poincaré rank two and one regular singularity.

4 Procedure of degeneration

In this section, we explain confluence of singularities of linear differential equations and the way how they induce the degeneration of Painlevé-type equations.

As an example, we treat the degeneration of $A_5^{(1)}$-type Fuji-Suzuki system to $A_5^{(1)}$-type Noumi-Yamada system. It corresponds to the confluence of singularities of $21, 21, 111, 111$-type Fuchsian system that leads to $(2)(1), 111, 111$-type equation.

Consider the following system of linear differential equations

$$\frac{dY}{dx} = A(x)Y, \tag{4.1}$$

$$A(x) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}, \tag{4.2}$$

whose Riemann scheme is given by

$$\begin{pmatrix} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ \theta_1^0 & 0 & 0 & \theta_2^\infty \\ \theta_2^0 & \theta^1 & \theta^t & \theta_3^\infty \end{pmatrix}.$$

Thus its spectral type is $21, 21, 111, 111$.

Now we put $t = 1 + \varepsilon\tilde{t}$ and take the limit $\varepsilon \to 0$ so that the singular point $x = t$ merges to $x = 1$. Suppose the coefficient matrix

$$A(x) = \frac{A_0}{x} + \frac{-\varepsilon\tilde{t}A_1}{(x-1)(x-1-\varepsilon\tilde{t})} + \frac{A_1 + A_t}{x-1-\varepsilon\tilde{t}} \tag{4.3}$$

tends to

$$\tilde{A}(x) = \frac{\tilde{A}_0^{(0)}}{x} + \frac{\tilde{A}_1^{(-1)}}{(x-1)^2} + \frac{\tilde{A}_1^{(0)}}{x-1} \tag{4.4}$$

as $\varepsilon \to 0$. Here the coefficient (4.4) corresponds to a system of linear differential equations with spectral type $(2)(1), 111, 111$, whose Riemann scheme is

$$\begin{pmatrix} x=0 & \overbrace{\quad x=1 \quad} & x=\infty \\ 0 & 0 \quad 0 & \tilde{\theta}_1^\infty \\ \tilde{\theta}_1^0 & 0 \quad 0 & \tilde{\theta}_2^\infty \\ \tilde{\theta}_2^0 & \tilde{t} \quad \tilde{\theta}^1 & \tilde{\theta}_3^\infty \end{pmatrix}.$$

By comparing (4.3) and (4.4), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-\varepsilon\tilde{t}\theta^1) = \lim_{\varepsilon\to 0}\mathrm{tr}(-\varepsilon\tilde{t}A_1) = \mathrm{tr}(\tilde{A}_1^{(-1)}) = \tilde{t}, \\
&\lim_{\varepsilon\to 0}(\theta^1 + \theta^t) = \lim_{\varepsilon\to 0}\mathrm{tr}(A_1 + A_t) = \mathrm{tr}(\tilde{A}_1^{(0)}) = \tilde{\theta}^1, \\
&\lim_{\varepsilon\to 0}(\theta_1^0 + \theta_2^0) = \lim_{\varepsilon\to 0}\mathrm{tr}(A_0) = \mathrm{tr}(\tilde{A}_0^{(0)}) = \tilde{\theta}_1^0 + \tilde{\theta}_2^0.
\end{aligned}$$

Take the above into account, we put the following relations:

$$\theta_i^0 = \tilde{\theta}_i^0 \quad (i = 1, 2), \quad \theta^1 = -\varepsilon^{-1}, \quad \theta^t = \tilde{\theta}^1 + \varepsilon^{-1}, \quad \theta_j^\infty = \tilde{\theta}_j^\infty \quad (j = 1, 2, 3).$$

Furthermore, we define a canonical transformation in the following manner:

$$\begin{aligned}
&q_1 = 1 + \varepsilon\tilde{t}\tilde{q}_1, \quad p_1 = \varepsilon^{-1}\tilde{t}^{-1}\tilde{p}_1, \quad q_2 = 1 + \varepsilon\tilde{t}\tilde{q}_2, \quad p_2 = \varepsilon^{-1}\tilde{t}^{-1}\tilde{p}_2, \\
&H_{FS}^{A_5} = \varepsilon^{-1}(\tilde{H} + t^{-1}(\tilde{p}_1\tilde{q}_1 + \tilde{p}_2\tilde{q}_2)).
\end{aligned}$$

It is easy to check that $\lim_{\varepsilon\to 0}\tilde{H} = H_{NY}^{A_5}$. In this way, we obtain $A_5^{(1)}$-type Noumi-Yamada system.

In the case of two dimensional phase space, the source equation of degeneration scheme, which governs deformation of Fuchsian equation, is the sixth Painlevé equation. Meanwhile, in the 4-dimensional case we

have four source equations, namely, the Garnier system in two variables, the $A_5^{(1)}$-type Fuji-Suzuki system, the $D_6^{(1)}$-type Sasano system, and the sixth matrix Painlevé system [30]. Accordingly, there are four series of degenerations.

The linear equation associated to the Garnier system has the same spectral type at each singular point, that is, 11. Thus the spectral type of its confluent equation is uniquely determined by the singularity pattern. For example, there is only one linear differential equation with three regular singular points and one irregular singular point of Poincaré rank one, which corresponds to $2+1+1+1$. Its spectral type is $(1)(1), 11, 11, 11$.

On the other hand, spectral types of linear equations associted to the remaining three equations, the Fuji-Suzuki system, the Sasano system, and the sixth matrix Painlevé system are $21, 21, 111, 111$, and $31, 31, 22, 1111$, and $22, 22, 22, 211$, respectively. Each equation includes singularities with different spectral types. In this case, it is not sufficient to give only the singularity patterns to specify confluent linear equations. Let us consider, for example, degenerate equations of the Fuji-Suzuki system corresponding to $2+1+1$. Since the spectral type of their source is $21, 21, 111, 111$, the confluence of 21 and 21 leads to $(2)(1), 111, 111$, the confluence of 111 and 111 leads to $(1)(1)(1), 21, 21$, and the confluence of 21 and 111 leads to $(11)(1), 21, 111$.

The figure in section 1 is the degeneration scheme of 4-dimensional Painlevé-type equations in terms of confluence of singularities of associated linear equations represented by their spectral types. The data on degenerations are given in the appendix.

As the closing of this section, we make some remarks on the degeneration scheme.

First, the correspondence of Painlevé-type equation to the linear equation is not one-to-one. For example, the deformation of $(2)(1), 111, 111$ and $(11)(11), 22, 31$ are governed by the same equation, the $A_5^{(1)}$-type Noumi-Yamada system. In fact, these two linear equations $(2)(1), 111, 111$ and $(11)(11), 22, 31$ are converted into each other by the Laplace transform. The Laplace transform will be discussed in section 6.

Secondly, it is impossible to merge two singular points without changing the number of accessory parameters unless one of the spectral types of singular points is a refinement of the other. Therefore, for example, the linear equation with only one irregular singularity (i.e. corresponds to 4) does not appear in the degeneration scheme of the $31, 22, 22, 1111$ system since singularity of type 22 and 31 cannot be merged. For the formula of rigidity index, see [34].

Finally, we would like to comment on the number of deformation parameters. Some degenerate $21, 21, 111, 111$ system admit 2-dimensional deformation. Those are $(1)(1)(1), 21, 21$, and $((1)(1))((1)), 21$, and $(2)(1), (1)(1)(1)$, and $(((1)(1)))(((1)))$ (see section 5 for details). As to those equations, we obtain Hamiltonians correspond to them not by confluence from $21, 21, 111, 111$, but from $(1)(1)(1), 21, 21$. Here the deformation equation of $(1)(1)(1), 21, 21$ in Hamiltonian form is calculated by the use of the technique described in an appendix of [14].

5 Isomonodromic deformation of linear equations

We will devide the linear equations into four families, and describe each of them in detail.

5.1 Garnier system and degenerate Garnier systems

In the first place, we will see a system of deformation equations associated with the Garnier system of two variables.

The Garnier system is obtained from isomonodromic deformation of a Fuchsian equation with five regular singular points. This Fuchsian equation is denoted by the singularity pattern $1+1+1+1+1$. Local data

that characterize linear equation are given by characteristic exponents at each singular point, and this type of Fuchsian equation can be reduced to a system with the following Riemann scheme by a suitable gauge transformation:

$$\left(\begin{matrix} x=0 & x=1 & x=t_1 & x=t_2 & x=\infty \\ 0 & 0 & 0 & 0 & \theta_1^\infty \\ \theta^0 & \theta^1 & \theta^{t_1} & \theta^{t_2} & \theta_2^\infty \end{matrix}\right).$$

The Fuchs relation is written as $\theta^0+\theta^1+\theta^{t_1}+\theta^{t_2}+\theta_1^\infty+\theta_2^\infty=0$. We will see a parameterization of the Fuchsian system. In this case the deformation system is simply written by using coefficients of the Fuchsian system.

$$\text{(5.1)}\qquad \begin{cases} \dfrac{\partial Y}{\partial x}=\left(\dfrac{A_0}{x}+\dfrac{A_1}{x-1}+\dfrac{A_{t_1}}{x-t_1}+\dfrac{A_{t_2}}{x-t_2}\right)Y, \\ \dfrac{\partial Y}{\partial t_1}=-\dfrac{A_{t_1}}{x-t_1}Y, \qquad \dfrac{\partial Y}{\partial t_2}=-\dfrac{A_{t_2}}{x-t_2}Y. \end{cases}$$

Here A_0, A_1, and A_t are given as follows:

$$A_\xi=\begin{pmatrix}1 & \\ & u\end{pmatrix}^{-1}P^{-1}\hat{A}_\xi P\begin{pmatrix}1 & \\ & u\end{pmatrix}, \quad (\xi=0,1,t_1,t_2),$$

$$\hat{A}_0=\begin{pmatrix}1\\0\end{pmatrix}\left(\theta^0,\ -1+\frac{q_1}{t_1}+\frac{q_2}{t_2}\right), \quad \hat{A}_1=\begin{pmatrix}1\\ p_1q_1+p_2q_2-\theta_2^\infty\end{pmatrix}\left(\theta^1+\theta_2^\infty-p_1q_1-p_2q_2,\ 1\right),$$

$$\hat{A}_{t_i}=\begin{pmatrix}1\\ t_ip_i\end{pmatrix}\left(\theta^{t_i}+p_iq_i,\ -\frac{q_i}{t_i}\right), \qquad \text{where } P=\begin{pmatrix}1 & 0\\ \frac{a}{\theta_1^\infty-\theta_2^\infty} & 1\end{pmatrix}.$$

Here a is the $(2,1)$-element of the matrix $\hat{A}_\infty:=-\hat{A}_0-\hat{A}_1-\hat{A}_{t_1}-\hat{A}_{t_2}$.

The compatibility conditions of these systems produce a Hamiltonain system. This is the partial differential system called Garnier system and the system is given by the following Hamiltonians:

$$\text{(5.2)}\qquad \begin{aligned} & t_i(t_i-1)H^{1+1+1+1+1}_{Gar,t_i}\left(\begin{matrix}\theta^0,\theta^1,\theta^{t_1},\theta^{t_2}\\ \theta_1^\infty,\theta_2^\infty\end{matrix};\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right) \\ &= t_i(t_i-1)H_{\mathrm{VI}}\left(\begin{matrix}\theta_2^\infty,\theta^1\\ \theta^{t_i},\theta^0+\theta^{t_{i+1}}+1\end{matrix};t_i;q_i,p_i\right)+(2q_ip_i+q_{i+1}p_{i+1}-\theta^1-2\theta_2^\infty)q_1q_2p_{i+1} \\ &\quad -\frac{1}{t_i-t_{i+1}}\{t_i(t_i-1)(p_iq_i+\theta^{t_i})p_iq_{i+1}-t_i(t_{i+1}-1)(2p_iq_i+\theta^{t_i})p_{i+1}q_{i+1} \\ &\quad +t_{i+1}(t_i-1)(p_{i+1}^2q_{i+1}+\theta^{t_{i+1}}(p_{i+1}-p_i))q_i\}, \qquad (i\in\mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

Furthermore the parameter u, which expresses gauge freedom, satisfy the following equations:

$$\text{(5.3)}\qquad t_1(t_1-1)\frac{1}{u}\frac{\partial u}{\partial t_1}=q_1\{2p_1(t_1-q_1)+\theta^1+2\theta_2^\infty\}-2q_1p_2q_2+t_1\theta^{t_1},$$

$$\text{(5.4)}\qquad t_2(t_2-1)\frac{1}{u}\frac{\partial u}{\partial t_2}=q_2\{2p_2(t_2-q_2)+\theta^1+2\theta_2^\infty\}-2q_2p_1q_1+t_2\theta^{t_2}.$$

In particular, time evolution of p_i,q_i $(i=1,2)$ is independent of u's behavior.

In the second place, we will see the non-Fuchsian system obtained by a confluence of two singular points from the above Fuchsian system. This system is expressed by the singularity pattern $2+1+1+1$.

Singularity pattern: 2+1+1+1

Riemann scheme is given by

$$\left(\begin{array}{cccc} x=0 & x=1 & x=t_2/t_1 & \overbrace{\quad x=\infty \quad} \\ 0 & 0 & 0 & 0 \quad \theta^\infty_1 \\ \theta^0 & \theta^1 & \theta^t & -t_1 \quad \theta^\infty_2 \end{array}\right),$$

and then the Fuchs-Hukuhara relation is written as $\theta^0+\theta^1+\theta^t+\theta^\infty_1+\theta^\infty_2=0$. The system of deformation equations are expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0}{x}+\dfrac{A_1}{x-1}+\dfrac{A_t}{x-\frac{t_2}{t_1}}+A_\infty\right)Y, \\ \dfrac{\partial Y}{\partial t_1} = \left(E_2 x + B_1 + \dfrac{\frac{t_2}{t_1^2}A_t}{x-\frac{t_2}{t_1}}\right)Y, \qquad \dfrac{\partial Y}{\partial t_2} = -\dfrac{\frac{1}{t_1}A_t}{x-\frac{t_2}{t_1}}Y. \end{cases} \tag{5.5}$$

Here

$$\begin{aligned} A_\xi &= \begin{pmatrix} 1 & \\ & u \end{pmatrix}^{-1} \hat{A}_\xi \begin{pmatrix} 1 & \\ & u \end{pmatrix}, \quad (\xi=0,1,t), \\ \hat{A}_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\mu_1\lambda_1+\mu_2\lambda_2-\theta^1-\theta^t-\theta^\infty_1,\ -\mu_1\lambda_1-\mu_2\lambda_2-\theta^\infty_2\right), \\ \hat{A}_1 &= \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \left(\theta^1-\mu_1\lambda_1,\ \mu_1\right), \quad \hat{A}_t = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \left(\theta^t-\mu_2\lambda_2,\ \mu_2\right), \quad A_\infty = \begin{pmatrix} 0 & \\ & t_1 \end{pmatrix}, \\ A^{(0)}_\infty &= -(A_0+A_1+A_t), \quad E_2 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}, \quad B_1 = \frac{1}{t_1}\begin{pmatrix} 0 & (-A^{(0)}_\infty)_{12} \\ (-A^{(0)}_\infty)_{21} & 0 \end{pmatrix}. \end{aligned}$$

The Hamiltonians are given as

$$\begin{aligned} &\tilde{H}^{2+1+1+1}_{Gar,t_1}\left(\begin{matrix} \theta^\infty_2, \theta^1 \\ \theta^t, -\theta^0-1 \end{matrix}; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} \lambda_1,\mu_1 \\ \lambda_2,\mu_2 \end{matrix}\right) \\ &= \tilde{H}_{\mathrm{V}}\left(\begin{matrix} \theta^0+\theta^\infty_2,\ \theta^0+\theta^t+\theta^\infty_1 \\ \theta^1 \end{matrix}; t_1; \lambda_1,\mu_1\right) + \frac{\mu_2\lambda_2}{t_1}(1-\lambda_1)(\mu_1-(\mu_1\lambda_1-\theta^1)) \\ &\quad + \frac{1}{t_1-t_2}(\mu_1(\lambda_1-\lambda_2)-\theta^1)(\mu_2(\lambda_2-\lambda_1)-\theta^t), \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\tilde{H}^{2+1+1+1}_{Gar,t_2}\left(\begin{matrix} \theta^\infty_2, \theta^1 \\ \theta^t, -\theta^0-1 \end{matrix}; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} \lambda_1,\mu_1 \\ \lambda_2,\mu_2 \end{matrix}\right) \\ &= \tilde{H}_{\mathrm{V}}\left(\begin{matrix} \theta^0+\theta^\infty_2,\ \theta^0+\theta^1+\theta^\infty_1 \\ \theta^t \end{matrix}; t_2; \lambda_2,\mu_2\right) + \frac{\mu_1\lambda_1}{t_2}(1-\lambda_2)(\mu_2-(\mu_2\lambda_2-\theta^t)) \\ &\quad + \frac{1}{t_2-t_1}(\mu_1(\lambda_1-\lambda_2)-\theta^1)(\mu_2(\lambda_2-\lambda_1)-\theta^t). \end{aligned} \tag{5.7}$$

Notice that $\tilde{H}_{\mathrm{V}}$ is different from H_{V} (see the beginning of Introduction).

When we change canonical variables as $\lambda_1 = 1-\frac{1}{q_1}$, $\mu_1 = q_1(p_1q_1-\theta^1)$, $\lambda_2 = 1-\frac{1}{q_2}$, and $\mu_2 = q_2(p_2q_2-\theta^t)$,

the Hamiltonians are given as

$$H^{2+1+1+1}_{Gar,t_1}\begin{pmatrix}\theta^\infty_2,\theta^1 & t_1 & q_1,p_1\\ \theta^t,-\theta^0-1 & t_2 & q_2,p_2\end{pmatrix} \tag{5.8}$$
$$= H_{\mathrm{V}}\begin{pmatrix}\theta^0+\theta^1+\theta^\infty_1,\ 2\theta^0+\theta^\infty_1+\theta^\infty_2 \\ -\theta^0-\theta^\infty_1\end{pmatrix};t_1;q_1,p_1\Bigg) + \frac{p_1}{t_1}(p_2q_2(q_2-1)+\theta^t(q_1-q_2))$$
$$-\frac{1}{t_1-t_2}(p_1(q_1-q_2)-\theta^1)(p_2(q_1-q_2)+\theta^t),$$

$$H^{2+1+1+1}_{Gar,t_2}\begin{pmatrix}\theta^\infty_2,\theta^1 & t_1 & q_1,p_1\\ \theta^t,-\theta^0-1 & t_2 & q_2,p_2\end{pmatrix} \tag{5.9}$$
$$= H_{\mathrm{V}}\left(\begin{matrix}\theta^0+\theta^t+\theta^\infty_1,\ 2\theta^0+\theta^\infty_1+\theta^\infty_2 \\ -\theta^0-\theta^\infty_1\end{matrix};t_2;q_2,p_2\right) + \frac{p_2}{t_2}(p_1q_1(q_1-1)+\theta^1(q_2-q_1))$$
$$-\frac{1}{t_2-t_1}(p_1(q_1-q_2)-\theta^1)(p_2(q_1-q_2)+\theta^t).$$

The gauge u satisfy the equations:

$$\frac{1}{u}\frac{\partial u}{\partial t_1} = \frac{1}{t_1}(p_1+\theta^\infty_1-\theta^\infty_2), \quad \frac{1}{u}\frac{\partial u}{\partial t_2} = \frac{p_2}{t_2}. \tag{5.10}$$

Singularity pattern: 3+1+1

When Riemann scheme is given as

$$\begin{pmatrix} x=0 & x=t_2-t_1 & \overbrace{\begin{matrix} & x=\infty & \end{matrix}} \\ 0 & 0 & \begin{matrix}0 & 0 & \theta^\infty_1\end{matrix} \\ \theta^0 & \theta^1 & \begin{matrix}-1 & t_2 & \theta^\infty_2\end{matrix}\end{pmatrix},$$

then the Fuchs-Hukuhara relation is written as $\theta^0+\theta^1+\theta^\infty_1+\theta^\infty_2=0$.

The system of deformation equations is given as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A^{(0)}_0}{x} + \dfrac{A^{(0)}_1}{x-(t_2-t_1)} + A^{(-1)}_\infty + A^{(-2)}_\infty x\right)Y, \\ \dfrac{\partial Y}{\partial t_1} = \dfrac{A^{(0)}_1}{x-(t_2-t_1)}Y, \qquad \dfrac{\partial Y}{\partial t_2} = \left(-\dfrac{A^{(0)}_1}{x-(t_2-t_1)} - A^{(-2)}_\infty x + B_1\right)Y. \end{cases} \tag{5.11}$$

Here

$$A^{(-k)}_\xi = \begin{pmatrix}1 & \\ & u\end{pmatrix}^{-1}\hat{A}_\xi\begin{pmatrix}1 & \\ & u\end{pmatrix},$$
$$\hat{A}^{(0)}_0 = \begin{pmatrix}q_2\\1\end{pmatrix}\left(p_2,\ -p_2q_2+\theta^0\right), \quad \hat{A}^{(0)}_1 = \begin{pmatrix}q_1\\1\end{pmatrix}\left(p_1,\ -p_1q_1+\theta^1\right), \quad A^{(-2)}_\infty = \begin{pmatrix}0 & \\ & 1\end{pmatrix},$$
$$\hat{A}^{(-1)}_\infty = -\begin{pmatrix}0 & p_1q_1+p_2q_2+\theta^\infty_1 \\ 1 & t_2\end{pmatrix}, \quad B_1 = \begin{pmatrix}0 & (-A^{(-1)}_\infty)_{12} \\ (-A^{(-1)}_\infty)_{21} & 0\end{pmatrix}.$$

The Hamiltonians are written as

$$H^{3+1+1}_{Gar,t_1}\begin{pmatrix}\theta^1,\theta^0 & t_1 & q_1,p_1 \\ \theta^\infty_1 & t_2 & q_2,p_2\end{pmatrix} \tag{5.12}$$
$$= H_{\mathrm{IV}}\left(\theta^1,\theta^\infty_1;t_1;q_1,p_1\right) + p_2q_2p_1 + \frac{1}{t_1-t_2}\{p_1(q_1-q_2)-\theta^1\}\{p_2(q_2-q_1)-\theta^0\},$$

$$H^{3+1+1}_{Gar,t_2}\begin{pmatrix}\theta^1,\theta^0 & t_1 & q_1,p_1 \\ \theta^\infty_1 & t_2 & q_2,p_2\end{pmatrix} \tag{5.13}$$
$$= H_{\mathrm{IV}}\left(\theta^0,\theta^\infty_1;t_2;q_2,p_2\right) + p_1q_1p_2 + \frac{1}{t_2-t_1}\{p_1(q_1-q_2)-\theta^1\}\{p_2(q_2-q_1)-\theta^0\}.$$

The gauge u satisfies

$$\frac{1}{u}\frac{\partial u}{\partial t_1}=-p_1,\quad \frac{1}{u}\frac{\partial u}{\partial t_2}=t_2-p_2. \tag{5.14}$$

Singularity pattern: 2+2+1

Rienmann scheme is given by

$$\left(\begin{array}{ccc} \overbrace{\begin{array}{cc} 0 & 0 \\ \frac{t_2}{t_1} & \theta^0 \end{array}}^{x=0} & \begin{array}{c} x=1 \\ 0 \\ \theta^1 \end{array} & \overbrace{\begin{array}{cc} 0 & \theta^\infty_1 \\ -t_1 & \theta^\infty_2 \end{array}}^{x=\infty} \end{array}\right),$$

and the Fuchs-Hukuhara relation is written as $\theta^0+\theta^1+\theta^\infty_1+\theta^\infty_2=0$.

The system of deformation equations is given as

$$\begin{cases} \dfrac{\partial Y}{\partial x}=\left(\dfrac{A_0^{(-1)}}{x^2}+\dfrac{A_0^{(0)}}{x}+\dfrac{A_1^{(0)}}{x-1}+A_\infty\right)Y, \\ \dfrac{\partial Y}{\partial t_1}=\left(E_2x+B_1+\dfrac{A_0^{(-1)}}{t_1x}\right)Y, \qquad \dfrac{\partial Y}{\partial t_2}=-\dfrac{A_0^{(-1)}}{t_2x}Y. \end{cases} \tag{5.15}$$

Here

$$\begin{aligned} A_\xi^{(-k)}&=\begin{pmatrix}1&\\&u\end{pmatrix}^{-1}\hat{A}_\xi\begin{pmatrix}1&\\&u\end{pmatrix}, \\ \hat{A}_0^{(-1)}&=\frac{t_2}{t_1}\begin{pmatrix}1\\1\end{pmatrix}(1-\mu_2,\ \mu_2), \quad \hat{A}_0^{(0)}=\begin{pmatrix}\mu_1\lambda_1-\theta^1-\theta^\infty_1 & -\mu_1\lambda_1-\mu_2\lambda_2-\theta^\infty_2 \\ \mu_1\lambda_1+(1-\mu_2)\lambda_2-\theta^1-\theta^\infty_1 & -\mu_1\lambda_1-\theta^\infty_2\end{pmatrix}, \\ \hat{A}_1^{(0)}&=\begin{pmatrix}1\\ \lambda_1\end{pmatrix}(-\mu_1\lambda_1+\theta^1,\ \mu_1), \quad A_\infty=\begin{pmatrix}0&\\&t_1\end{pmatrix}, \quad A_\infty^{(0)}=-(A_0^{(0)}+A_1^{(0)}), \\ E_2&=\begin{pmatrix}0&\\&1\end{pmatrix}, \quad B_1=\frac{1}{t_1}\begin{pmatrix}0 & (-A_\infty^{(0)})_{12} \\ (-A_\infty^{(0)})_{21} & 0\end{pmatrix}. \end{aligned}$$

The Hamiltonians are expressed as

$$\begin{aligned} &t_1\tilde{H}^{2+2+1}_{Gar,t_1}\begin{pmatrix}\theta^0,\theta^1 & t_1 & \lambda_1,\mu_1 \\ \theta^\infty_1,\theta^\infty_2 ; & t_2 ; & \lambda_2,\mu_2\end{pmatrix} \\ &=t_1\tilde{H}_{\mathrm{V}}\begin{pmatrix}\theta^0+\theta^\infty_2,\theta^0+\theta^\infty_1 \\ \theta^1\end{pmatrix};t_1;\lambda_1,\mu_1\Big)+((\mu_1\lambda_1-\theta^1)\lambda_1-\mu_1)\mu_2\lambda_2+\mu_1\lambda_2 \\ &\quad-\frac{t_2}{t_1}(\mu_1(\lambda_1-1)-\theta^1)(\mu_2(\lambda_1-1)+1), \end{aligned} \tag{5.16}$$

$$\begin{aligned} &t_2\tilde{H}^{2+2+1}_{Gar,t_2}\begin{pmatrix}\theta^0,\theta^1 & t_1 & \lambda_1,\mu_1 \\ \theta^\infty_1,\theta^\infty_2 ; & t_2 ; & \lambda_2,\mu_2\end{pmatrix} \\ &=t_2H_{\mathrm{III}}(D_6)\left(\theta^\infty_2,-\theta^0;t_2;\lambda_2,\mu_2\right)-\mu_1\lambda_1\lambda_2+\frac{t_2}{t_1}(\mu_1(\lambda_1-1)-\theta^1)(\mu_2(\lambda_1-1)+1). \end{aligned} \tag{5.17}$$

If we change the canonical variables as

$$\lambda_1=1-\frac{1}{q_1},\quad \mu_1=q_1(p_1q_1-\theta^1),$$

then we obtain

$$
\begin{aligned}
(5.18)\qquad & t_1 H^{2+2+1}_{Gar,t_1}\begin{pmatrix} \theta^1,\theta^0 & ; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} q_1,p_1 \\ q_2,p_2 \end{matrix} \\ -\theta^0-\theta^1-\theta^\infty_2 & \end{pmatrix} \\
&= t_1 H_{\mathrm{V}}\begin{pmatrix} -\theta^\infty_2,\theta^0-\theta^1 & ;t_1;q_1,p_1 \\ \theta^1+\theta^\infty_2 & \end{pmatrix} + q_1q_2(p_1q_1-\theta^1)+p_2q_2(\theta^1+p_1-2p_1q_1)-\frac{t_2}{t_1}p_1(p_2-q_1),
\end{aligned}
$$

$$
\begin{aligned}
(5.19)\qquad & t_2 H^{2+2+1}_{Gar,t_2}\begin{pmatrix} \theta^1,\theta^0 & ; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} q_1,p_1 \\ q_2,p_2 \end{matrix} \\ -\theta^0-\theta^1-\theta^\infty_2 & \end{pmatrix} \\
&= t_2 H_{\mathrm{III}}(D_6)\left(\theta^\infty_2,-\theta^0;t_2;q_2,p_2\right)-(p_1q_1-\theta^1)q_2(q_1-1)+\frac{t_2}{t_1}p_1(p_2-q_1).
\end{aligned}
$$

The gauge u satisfies

$$
(5.20)\qquad \frac{1}{u}\frac{\partial u}{\partial t_1}=\frac{p_1+\theta^\infty_1-\theta^\infty_2}{t_1},\quad \frac{1}{u}\frac{\partial u}{\partial t_2}=-\frac{q_2}{t_2}.
$$

Singularity pattern: 3+2

Riemann scheme is given by

$$
\begin{pmatrix} \overbrace{\begin{matrix} 0 & 0 \\ t_1 & \theta^0 \end{matrix}}^{x=0} & \overbrace{\begin{matrix} 0 & 0 & \theta^\infty_1 \\ -1 & t_2 & \theta^\infty_2 \end{matrix}}^{x=\infty} \end{pmatrix},
$$

and then the Fuchs-Hukuhara relation is written as $\theta^0+\theta^\infty_1+\theta^\infty_2=0$.

The system of deformation equations is expressed as

$$
(5.21)\qquad \begin{cases} \dfrac{\partial Y}{\partial x}=\left(\dfrac{A^{(-1)}_0}{x^2}+\dfrac{A^{(0)}_0}{x}+A^{(-1)}_\infty+A^{(-2)}_\infty x\right)Y, \\ \dfrac{\partial Y}{\partial t_1}=-\dfrac{A^{(-1)}_0}{t_1x}Y,\qquad \dfrac{\partial Y}{\partial t_2}=(-A^{(-2)}_\infty x+B_1)Y, \end{cases}
$$

where

$$
\begin{aligned}
A^{(-k)}_\xi &= \begin{pmatrix} 1 & \\ & u \end{pmatrix}^{-1}\hat{A}_\xi\begin{pmatrix} 1 & \\ & u \end{pmatrix}, \\
\hat{A}^{(-1)}_0 &= \begin{pmatrix} q_2 \\ 1 \end{pmatrix}\left(-q_1,\ q_1q_2+t_1\right),\quad \hat{A}^{(0)}_0=\begin{pmatrix} -p_1q_1+p_2q_2 & -q_2(p_2q_2-\theta^0)+p_1(2q_1q_2+t_1) \\ p_2 & p_1q_1-p_2q_2+\theta^0 \end{pmatrix}, \\
A^{(-2)}_\infty &= \begin{pmatrix} 0 & \\ & 1 \end{pmatrix},\quad \hat{A}^{(-1)}_\infty=\begin{pmatrix} 0 & p_1q_1-p_2q_2-\theta^\infty_1 \\ -1 & -t_2 \end{pmatrix},\quad B_1=\begin{pmatrix} 0 & (-A^{(-1)}_\infty)_{12} \\ (-A^{(-1)}_\infty)_{21} & 0 \end{pmatrix}.
\end{aligned}
$$

The Hamiltonians are given by

$$
\begin{aligned}
&(5.22) \\
&H^{3+2}_{Gar,t_1}\left(\theta^0,\theta^\infty_1;\begin{matrix} t_1 \\ t_2 \end{matrix};\begin{matrix} q_1,p_1 \\ q_2,p_2 \end{matrix}\right)=H_{\mathrm{III}}(D_6)\left(-\theta^\infty_1,\theta^0+1;t_1;q_1,p_1\right)-p_1-\frac{q_1q_2}{t_1}(q_2-p_2+t_2)+p_1p_2-q_2, \\
&(5.23) \\
&H^{3+2}_{Gar,t_2}\left(\theta^0,\theta^\infty_1;\begin{matrix} t_1 \\ t_2 \end{matrix};\begin{matrix} q_1,p_1 \\ q_2,p_2 \end{matrix}\right)=H_{\mathrm{IV}}\left(\theta^0,\theta^\infty_1;t_2;q_2,p_2\right)-p_1q_1(p_2-2q_2-t_2)-q_1q_2+t_1p_1.
\end{aligned}
$$

The gauge u satisfies

$$
(5.24)\qquad \frac{1}{u}\frac{\partial u}{\partial t_1}=-\frac{q_1}{t_1},\quad \frac{1}{u}\frac{\partial u}{\partial t_2}=t_2-p_2.
$$

Singularity pattern: 4+1

Riemann scheme is given by

$$\left(\begin{array}{ccccc} x=0 & \multicolumn{4}{c}{\overbrace{\qquad\qquad x=\infty \qquad\qquad}} \\ 0 & 0 & 0 & 0 & \theta^\infty_1 \\ \theta^0 & 1 & 2t_2 & t_1+t_2^2 & \theta^\infty_2 \end{array}\right),$$

and then the Fuchs-Hukuhara relation is written as $\theta^0+\theta^\infty_1+\theta^\infty_2=0$.

The system of deformation equations is expressed as

$$\left\{\begin{aligned} \frac{\partial Y}{\partial x} &= \left(\frac{A_0^{(0)}}{x}+A_\infty^{(-1)}+A_\infty^{(-2)}x+A_\infty^{(-3)}x^2\right)Y, \\ \frac{\partial Y}{\partial t_1} &= (-E_2x+B_1)Y, \qquad \frac{\partial Y}{\partial t_2} = \left(-E_2x^2+A_\infty^{(-2)}x+A_\infty^{(-1)}+T_1^{(\infty)}\right)Y. \end{aligned}\right. \tag{5.25}$$

Here

$$A_\xi^{(-k)}=\begin{pmatrix}1 & \\ & u\end{pmatrix}^{-1}\hat{A}_\xi^{(-k)}\begin{pmatrix}1 & \\ & u\end{pmatrix}, \qquad \hat{A}_0^{(0)}=\begin{pmatrix}q_2\\1\end{pmatrix}\left(p_2,\ -p_2q_2+\theta^0\right),$$

$$A_\infty^{(-3)}=\begin{pmatrix}0 & \\ & -1\end{pmatrix}, \quad \hat{A}_\infty^{(-2)}=\begin{pmatrix}0 & p_1\\ 1 & -2t_2\end{pmatrix}, \quad \hat{A}_\infty^{(-1)}=\begin{pmatrix}-p_1 & p_1(q_1+t_2)-p_2q_2-\theta^\infty_1\\ -q_1+t_2 & p_1-t_1-t_2^2\end{pmatrix},$$

$$B_1=\begin{pmatrix}0 & \left(A_\infty^{(-2)}\right)_{12}\\ \left(A_\infty^{(-2)}\right)_{21} & 0\end{pmatrix}, \quad T_1^{(\infty)}=\begin{pmatrix}0 & \\ & t_1+t_2^2\end{pmatrix}, \quad E_2=\begin{pmatrix}0 & \\ & 1\end{pmatrix}.$$

The Hamiltonians are given by

$$H^{4+1}_{Gar,t_1}\left(\theta^0,\theta^\infty_1;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)=H_{\mathrm{II}}\left(-\theta^\infty_1;t_1;q_1,p_1\right)+p_2q_2(q_1-q_2+t_2)+p_1p_2+\theta^0q_2, \tag{5.26}$$

$$\begin{aligned} &H^{4+1}_{Gar,t_2}\left(\theta^0,\theta^\infty_1;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right)\\ &=-p_2^2q_2-t_2p_2q_2^2+t_2^2p_2q_2+\theta^0t_2q_2-\theta^\infty_1p_2+p_1p_2(q_1-2q_2+t_2)+q_1q_2(p_2q_2-\theta^0)+\theta^0p_1+t_1p_2q_2. \end{aligned} \tag{5.27}$$

The gauge satisfies

$$\frac{1}{u}\frac{\partial u}{\partial t_1}=-q_1-t_2, \quad \frac{1}{u}\frac{\partial u}{\partial t_2}=p_2-t_1-t_2^2. \tag{5.28}$$

Singularity pattern: 5

Riemann scheme is given by

$$\left(\begin{array}{ccccc} \multicolumn{5}{c}{\overbrace{\qquad\qquad x=\infty \qquad\qquad}} \\ 0 & 0 & 0 & 0 & \theta^\infty_1 \\ -1 & 0 & -2t_2 & -t_1 & \theta^\infty_2 \end{array}\right),$$

and Fuchs-Hukuhara relation is written as $\theta^\infty_1+\theta^\infty_2=0$.

The system of deformation equations are expressed as

$$\left\{\begin{aligned} \frac{\partial Y}{\partial x} &= \left(A_\infty^{(-4)}x^3+A_\infty^{(-3)}x^2+A_\infty^{(-2)}x+A_\infty^{(-1)}\right)Y, \\ \frac{\partial Y}{\partial t_1} &= \left(A_\infty^{(-4)}x+A_\infty^{(-3)}\right)Y, \qquad \frac{\partial Y}{\partial t_2} = \left(A_\infty^{(-4)}x^2+A_\infty^{(-3)}x+A_\infty^{(-2)}+T_\infty^{(-2)}\right)Y, \end{aligned}\right. \tag{5.29}$$

where

$$A_\infty^{(-k)} = \begin{pmatrix} 1 & \\ & u \end{pmatrix}^{-1} \hat{A}_\infty^{(-k)} \begin{pmatrix} 1 & \\ & u \end{pmatrix},$$

$$A_\infty^{(-4)} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}, \quad \hat{A}_\infty^{(-3)} = \begin{pmatrix} 0 & q_2 \\ -1 & 0 \end{pmatrix}, \quad \hat{A}_\infty^{(-2)} = \begin{pmatrix} -q_2 & -p_1 \\ -q_1 & q_2 + 2t_2 \end{pmatrix},$$

$$\hat{A}_\infty^{(-1)} = \begin{pmatrix} p_1 - q_1 q_2 & p_1 q_1 - (p_2 - q_2 - 2t_2) q_2 - \theta_1^\infty \\ -p_2 & -p_1 + q_1 q_2 + t_1 \end{pmatrix}, \quad T_\infty^{(-2)} = \begin{pmatrix} 0 & \\ & -2t_2 \end{pmatrix}.$$

The Hamiltonians are given by

$$H^5_{Gar,t_1}\left(\theta_1^\infty; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix}\right) = -q_1(q_1 p_1 - \theta_1^\infty) + q_2(q_1(p_2 + q_2) - 2p_1 + t_1) + p_1(p_2 - 2t_2), \tag{5.30}$$

$$H^5_{Gar,t_2}\left(\theta_1^\infty; \begin{matrix} t_1 \\ t_2 \end{matrix}; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix}\right) = H_{\mathrm{IV}}\left(-1, \theta_1^\infty; 2t_2; q_2, p_2\right) + q_1 q_2(q_1 q_2 - 2p_1 + t_1) + p_1(p_1 - p_2 q_1 - t_1). \tag{5.31}$$

The gauge satisfies

$$\frac{1}{u}\frac{\partial u}{\partial t_1} = -q_1, \quad \frac{1}{u}\frac{\partial u}{\partial t_2} = 2t_2 - p_2. \tag{5.32}$$

5.2 Fuji-Suzuki system

Singularity pattern: 1+1+1+1

Spectral type: 21,21,111,111

Riemann scheme is given by

$$\begin{pmatrix} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ \theta_1^0 & 0 & 0 & \theta_2^\infty \\ \theta_2^0 & \theta^1 & \theta^t & \theta_3^\infty \end{pmatrix},$$

and then the Fuchs relation is written as $\theta_1^0 + \theta_2^0 + \theta^1 + \theta^t + \theta_1^\infty + \theta_2^\infty + \theta_3^\infty = 0$.

The system of deformation equations is expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0}{x} + \dfrac{A_1}{x-1} + \dfrac{A_t}{x-t}\right) Y, \\ \dfrac{\partial Y}{\partial t} = -\dfrac{A_t}{x-t} Y. \end{cases} \tag{5.33}$$

Here

$$A_\xi = U^{-1} P^{-1} \hat{A}_\xi P U, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a}{\theta_1^\infty - \theta_2^\infty} & 1 & 0 \\ \frac{1}{\theta_1^\infty - \theta_3^\infty}\left(b + \frac{ac}{\theta_1^\infty - \theta_2^\infty}\right) & \frac{c}{\theta_2^\infty - \theta_3^\infty} & 1 \end{pmatrix}, \quad U = \mathrm{diag}(1, u, v), \quad (\xi = 0, 1, t),$$

$$\hat{A}_0 = \begin{pmatrix} \theta_1^0 & \frac{q_1}{t} - 1 & \frac{q_2}{t} - 1 \\ 0 & \theta_2^0 & p_1(q_2 - q_1) + \theta_2^\infty + \theta_2^0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}_t = \begin{pmatrix} 1 \\ tp_1 \\ tp_2 \end{pmatrix} \left(\theta^t + p_1 q_1 + p_2 q_2,\ -\frac{q_1}{t},\ -\frac{q_2}{t}\right),$$

$$\hat{A}_1 = \begin{pmatrix} 1 \\ p_1 q_1 - \theta_2^\infty - \theta_2^0 \\ p_2 q_2 - \theta_3^\infty \end{pmatrix} \left(-p_1 q_1 - p_2 q_2 - \theta_1^0 - \theta^t - \theta_1^\infty,\ 1,\ 1\right),$$

$$\begin{cases} a = -tp_1(p_1q_1 + p_2q_2 + \theta^t) + (p_1q_1 - \theta^0_2 - \theta^\infty_2)(p_1q_1 + p_2q_2 + \theta^0_1 + \theta^t + \theta^\infty_1), \\ b = -tp_2(p_1q_1 + p_2q_2 + \theta^t) + (p_2q_2 - \theta^\infty_3)(p_1q_1 + p_2q_2 + \theta^0_1 + \theta^t + \theta^\infty_1), \\ c = p_2(q_1 - q_2) + \theta^\infty_3. \end{cases}$$

The Hamiltonian is given by

$$\tag{5.34} \begin{aligned} &H^{A_5}_{FS}\left(\begin{matrix}\theta^0_2+\theta^\infty_2, \theta^\infty_3, \theta^t \\ \theta^1, \theta^0_1, \theta^0_2\end{matrix}; t; \begin{matrix} q_1, p_1 \\ q_2, p_2\end{matrix}\right) \\ &= H_{\mathrm{VI}}\left(\begin{matrix}\theta^0_2+\theta^\infty_2, \theta^1+\theta^\infty_3 \\ \theta^t+\theta^\infty_3, \theta^0_1-\theta^0_2+1\end{matrix}; t; q_1, p_1\right) + H_{\mathrm{VI}}\left(\begin{matrix}\theta^\infty_3, \theta^0_2+\theta^1+\theta^\infty_2 \\ \theta^0_2+\theta^t+\theta^\infty_2, \theta^0_1-\theta^0_2-\theta^\infty_2+1\end{matrix}; t; q_2, p_2\right) \\ &+ \frac{1}{t(t-1)}(q_1-t)(q_2-1)\{(p_1q_1-\theta^0_2-\theta^\infty_2)p_2 + p_1(p_2q_2-\theta^\infty_3)\}. \end{aligned}$$

The gauge parameters u, v satisfy

$$\tag{5.35} t(t-1)\frac{1}{u}\frac{du}{dt} = 2p_1q_1(t-q_1) + p_2q_2(t-q_2) + (-\theta^0_1+\theta^0_2-\theta^t-\theta^\infty_1+\theta^\infty_2)q_1 + \theta^\infty_3 q_2 + q_1p_2(1-q_2) + t\theta^t,$$

$$\tag{5.36} t(t-1)\frac{1}{v}\frac{dv}{dt} = q_1\{p_1(t-q_1)+\theta^0_2+\theta^\infty_2\} + q_2\{2p_2(t-q_2)-\theta^0_1-\theta^t-\theta^\infty_1+\theta^\infty_3\} + p_1q_2(t-q_1) + t\theta^t.$$

Singularity pattern: 2+1+1

Spectral type: (2)(1),111,111

Riemann scheme is given by

$$\begin{pmatrix} x=0 & \overbrace{x=1}^{} & x=\infty \\ 0 & 0 \quad 0 & \theta^\infty_1 \\ \theta^0_1 & 0 \quad 0 & \theta^\infty_2 \\ \theta^0_2 & t \quad \theta^1 & \theta^\infty_3 \end{pmatrix},$$

and the Fuchs-Hukuhara relation is written as $\theta^0_1 + \theta^0_2 + \theta^1 + \theta^\infty_1 + \theta^\infty_2 + \theta^\infty_3 = 0$.

The system of deformation equations is expressed as

$$\tag{5.37} \begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A^{(-1)}_1}{(x-1)^2} + \dfrac{A^{(0)}_1}{x-1} + \dfrac{A^{(0)}_0}{x}\right)Y, \\ \dfrac{\partial Y}{\partial t} = -\dfrac{1}{x-1}\left(\dfrac{A^{(-1)}_1}{t}\right)Y. \end{cases}$$

Here

$$A^{(k)}_\xi = U^{-1}P^{-1}\hat{A}^{(k)}_\xi PU, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a}{\theta^\infty_1-\theta^\infty_2} & 1 & 0 \\ \frac{1}{\theta^\infty_1-\theta^\infty_3}\left(b+\frac{ac}{\theta^\infty_1-\theta^\infty_2}\right) & \frac{c}{\theta^\infty_2-\theta^\infty_3} & 1 \end{pmatrix}, \quad U = \mathrm{diag}(1, u, v),$$

$$\hat{A}^{(0)}_0 = \begin{pmatrix} \theta^0_1 & t(q_2-1) & t(q_1-1) \\ 0 & \theta^0_2 & p_2(q_1-q_2)+\theta^0_2+\theta^\infty_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}^{(-1)}_1 = \begin{pmatrix} 1 \\ p_2/t \\ p_1/t \end{pmatrix}(p_1+p_2+t, -t, -t),$$

$$\hat{A}^{(0)}_1 = \begin{pmatrix} \theta^0_2+\theta^1+\theta^\infty_2+\theta^\infty_3 & t(1-q_2) & t(1-q_1) \\ -a & -\theta^0_2-\theta^\infty_2 & p_2(q_2-q_1)-\theta^0_2-\theta^\infty_2 \\ -b & p_1(q_1-q_2)-\theta^\infty_3 & -\theta^\infty_3 \end{pmatrix},$$

$$\begin{cases} ta = p_2(q_2-1)(p_1+p_2+t) - (\theta_2^0+\theta_2^\infty)(p_1+t) - (2\theta_2^0+\theta^1+2\theta_2^\infty+\theta_3^\infty)p_2, \\ tb = p_1(q_1-1)(p_1+p_2+t) - \theta_3^\infty(p_2+t) - (\theta_2^0+\theta^1+\theta_2^\infty+2\theta_3^\infty)p_1, \\ c = p_1(q_2-q_1)+\theta_3^\infty. \end{cases}$$

The Hamiltonian is given by

$$\begin{aligned} &tH_{NY}^{A_5}\begin{pmatrix} \theta_1^\infty-\theta_3^\infty-1, \theta_3^\infty, -\theta_2^\infty & ;t; & q_1,p_1 \\ \theta_2^0+\theta_2^\infty, \theta_1^0-\theta_2^0 & & q_2,p_2 \end{pmatrix} \\ &= tH_{\mathrm{V}}\begin{pmatrix} \theta_1^\infty-1, \theta_1^0-\theta_2^0+\theta_1^\infty-\theta_2^\infty-\theta_3^\infty-1 & ;t;q_1,p_1 \\ -\theta_1^\infty+\theta_3^\infty+1 & \end{pmatrix} \\ &\quad + tH_{\mathrm{V}}\begin{pmatrix} \theta_2^0+\theta_1^\infty-\theta_3^\infty-1, \theta_1^0-\theta_2^0+\theta_1^\infty-\theta_2^\infty-\theta_3^\infty-1 & ;t;q_2,p_2 \\ -\theta_1^\infty+\theta_2^\infty+\theta_3^\infty+1 & \end{pmatrix} + 2p_1p_2q_1(q_2-1). \end{aligned} \tag{5.38}$$

The gauge parameters satisfy

$$t\frac{1}{u}\frac{du}{dt} = p_1(1-2q_1)+(2p_2+t)(1-q_2)-\theta_1^0+\theta_2^0-\theta_1^\infty+\theta_2^\infty+\theta_3^\infty, \tag{5.39}$$

$$t\frac{1}{v}\frac{dv}{dt} = (2p_1+t)(1-q_1)+2p_2(1-q_2)-\theta_1^0+\theta_2^0-\theta_1^\infty+\theta_2^\infty+\theta_3^\infty. \tag{5.40}$$

Spectral type: (11)(1),21,111

Riemann scheme is given by

$$\left(\begin{array}{cccc} \multicolumn{2}{c}{\overbrace{x=0}} & x=1 & x=\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ 0 & \theta_1^0 & 0 & \theta_2^\infty \\ t & \theta_2^0 & \theta^1 & \theta_3^\infty \end{array}\right),$$

and the Fuchs-Hukuhara relation is written as $\theta_1^0+\theta_2^0+\theta^1+\theta_1^\infty+\theta_2^\infty+\theta_3^\infty=0$.

The system of deformation equations is expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0^{(-1)}}{x^2}+\dfrac{A_0^{(0)}}{x}+\dfrac{A_1^{(0)}}{x-1}\right)Y, \\ \dfrac{\partial Y}{\partial t} = -\dfrac{1}{x}\left(\dfrac{A_0^{(-1)}}{t}\right)Y. \end{cases} \tag{5.41}$$

Here

$$A_\xi^{(k)} = U^{-1}P^{-1}\hat{A}_\xi^{(k)}PU, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a}{\theta_1^\infty-\theta_2^\infty} & 1 & 0 \\ \frac{1}{\theta_1^\infty-\theta_3^\infty}\left(b+\frac{ac}{\theta_1^\infty-\theta_2^\infty}\right) & \frac{c}{\theta_2^\infty-\theta_3^\infty} & 1 \end{pmatrix}, \quad U = \mathrm{diag}(1,u,v),$$

$$\hat{A}_1^{(0)} = \begin{pmatrix} 1 \\ -p_1q_1 \\ -p_2q_2 \end{pmatrix}(p_1q_1+p_2q_2+\theta^1, 1, 1), \quad \hat{A}_0^{(-1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(t, -1/q_1, -1/q_2),$$

$$\hat{A}_0^{(0)} = \begin{pmatrix} -p_1q_1-p_2q_2-\theta^1-\theta_1^\infty & -1 & -1 \\ -tq_1(p_1q_1-\theta_1^0-\theta_2^\infty) & p_1q_1-\theta_2^\infty & p_1q_1 \\ -tq_2(p_2q_2-\theta_3^\infty) & q_2(p_2q_2-\theta_3^\infty)/q_1 & p_2q_2-\theta_3^\infty \end{pmatrix},$$

$$\begin{cases} a = tq_1(p_1q_1-\theta_1^0-\theta_2^\infty)+p_1q_1(p_1q_1+p_2q_2+\theta^1), \\ b = tq_2(p_2q_2-\theta_3^\infty)+p_2q_2(p_1q_1+p_2q_2+\theta^1), \\ c = -q_2(p_2q_2-\theta_3^\infty)\left(\dfrac{1}{q_1}-\dfrac{1}{q_2}\right)+\theta_3^\infty. \end{cases}$$

The Hamiltonian is given by

$$
\begin{aligned}
(5.42)\qquad & tH_{FS}^{A_4}\begin{pmatrix}\theta^1,\theta_1^0,\theta_1^\infty & ;t; & q_1,p_1\\ \theta_2^\infty,\theta_3^\infty & & q_2,p_2\end{pmatrix}\\
&= tH_{\mathrm{V}}\begin{pmatrix}\theta_1^0+\theta^1+\theta_2^\infty+\theta_3^\infty,\theta^1+\theta_1^\infty-\theta_2^\infty-1 & ;t;q_1,p_1\\ -\theta^1-\theta_3^\infty & \end{pmatrix}\\
&\quad + tH_{\mathrm{V}}\begin{pmatrix}\theta^1+\theta_3^\infty,\theta^1+\theta_1^\infty-\theta_3^\infty-1 & ;t;q_2,p_2\\ -\theta^1 & \end{pmatrix}+p_1(q_2-1)\{p_2(q_1+q_2)-\theta_3^\infty\}.
\end{aligned}
$$

The gauge parameters satisfy

$$
(5.43)\quad tq_1\frac{1}{u}\frac{du}{dt}=q_2(p_2q_2-p_2-\theta_3^\infty)-q_1(2p_1+p_2+t)-\theta^1,\quad tq_2\frac{1}{v}\frac{dv}{dt}=-p_1(q_1+q_2)-(2p_2+t)q_2-\theta^1.
$$

Spectral type: (1)(1)(1),21,21

Riemann scheme is given by

$$
\begin{pmatrix}
x=0 & x=1 & \multicolumn{2}{c}{\overbrace{\quad x=\infty\quad}}\\
0 & 0 & 0 & \theta_1^\infty\\
0 & 0 & -t_1 & \theta_2^\infty\\
\theta^0 & \theta^1 & -t_2 & \theta_3^\infty
\end{pmatrix},
$$

and the Fuchs-Hukuhara relation is written as $\theta^0+\theta^1+\theta_1^\infty+\theta_2^\infty+\theta_3^\infty=0$.

The system of deformation equations is expressed as

$$
(5.44)\qquad \begin{cases}\dfrac{\partial Y}{\partial x}=\left(A_\infty+\dfrac{A_0^{(0)}}{x}+\dfrac{A_1^{(0)}}{x-1}\right)Y,\\[2mm] \dfrac{\partial Y}{\partial t_1}=(E_2x+B_1)Y,\qquad \dfrac{\partial Y}{\partial t_2}=(E_3x+B_2)Y,\end{cases}
$$

where

$$
A_\xi^{(k)}=U^{-1}\hat{A}_\xi^{(k)}U,\qquad B_i=U^{-1}\hat{B}_iU,\quad U=\mathrm{diag}(1,u,v),
$$

$$
A_\infty=\begin{pmatrix}0&&\\&t_1&\\&&t_2\end{pmatrix},\qquad \hat{A}_0^{(0)}=\begin{pmatrix}1\\ \mu_1\\ \mu_2\end{pmatrix}(\mu_1\lambda_1+\mu_2\lambda_2+\theta^0,\ -\lambda_1,\ -\lambda_2),
$$

$$
\hat{A}_1^{(0)}=\begin{pmatrix}1\\ \mu_1\lambda_1-\theta_2^\infty\\ \mu_2\lambda_2-\theta_3^\infty\end{pmatrix}(-\mu_1\lambda_1-\mu_2\lambda_2+\theta^1+\theta_2^\infty+\theta_3^\infty,\ 1,\ 1),\quad E_2=\mathrm{diag}(0,1,0),\quad E_3=\mathrm{diag}(0,0,1),
$$

$$
\hat{B}_1=\begin{pmatrix}0 & \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{12}}{t_1} & 0\\ \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{21}}{t_1} & 0 & \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{23}}{t_1-t_2}\\ 0 & \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{32}}{t_1-t_2} & 0\end{pmatrix},\quad \hat{B}_2=\begin{pmatrix}0 & 0 & \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{13}}{t_2}\\ 0 & 0 & \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{23}}{t_2-t_1}\\ \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{31}}{t_2} & \frac{(\hat{A}_0^{(0)}+\hat{A}_1^{(0)})_{32}}{t_2-t_1} & 0\end{pmatrix}.
$$

The Hamiltonian is given by

$$t_1\tilde{H}^{2+1+1+1}_{Gar,t_1}\begin{pmatrix}\theta^0,\theta^1 & t_1 & \lambda_1,\mu_1\\ \theta^\infty_2,\theta^\infty_3 ; & t_2 ; & \lambda_2,\mu_2\end{pmatrix} \tag{5.45}$$
$$= t_1\tilde{H}_{\mathrm{V}}\begin{pmatrix}\theta^0,\theta^1+\theta^\infty_3 & \\ \theta^\infty_2 & ;t_1;\lambda_1,\mu_1\end{pmatrix} + (1-\lambda_1)\mu_2\lambda_2(\mu_1-\mu_1\lambda_1+\theta^\infty_2)$$
$$+\frac{t_1}{t_1-t_2}(\mu_1(\lambda_1-\lambda_2)-\theta^\infty_2)(\mu_2(\lambda_2-\lambda_1)-\theta^\infty_3),$$

$$t_2\tilde{H}^{2+1+1+1}_{Gar,t_2}\begin{pmatrix}\theta^0,\theta^1 & t_1 & \lambda_1,\mu_1\\ \theta^\infty_1,\theta^\infty_2,\theta^\infty_3 ; & t_2 ; & \lambda_2,\mu_2\end{pmatrix} \tag{5.46}$$
$$= t_2\tilde{H}_{\mathrm{V}}\begin{pmatrix}\theta^0,\theta^1+\theta^\infty_2 & \\ \theta^\infty_3 & ;t_2;\lambda_2,\mu_2\end{pmatrix} + (1-\lambda_2)\mu_1\lambda_1(\mu_2-\mu_2\lambda_2+\theta^\infty_3)$$
$$+\frac{t_2}{t_2-t_1}(\mu_1(\lambda_1-\lambda_2)-\theta^\infty_2)(\mu_2(\lambda_2-\lambda_1)-\theta^\infty_3).$$

When we change the canonical variables as

$$\lambda_1 = 1-\frac{1}{q_1},\ \mu_1 = q_1(p_1q_1-\theta^\infty_2),\ \lambda_2 = 1-\frac{1}{q_2},\ \mu_2 = q_2(p_2q_2-\theta^\infty_3),$$

then we obtain

$$H^{2+1+1+1}_{Gar,\ t_1}\begin{pmatrix}\theta^0+\theta^\infty_1,\theta^\infty_2 & t_1 & q_1,p_1\\ \theta^\infty_3,\theta^\infty_1-1 ; & t_2 ; & q_2,p_2\end{pmatrix} = H_{\mathrm{V}}\begin{pmatrix}\theta^1+\theta^\infty_2,\theta^0+\theta^1 & \\ -\theta^1 & ;t_1;q_1,p_1\end{pmatrix} \tag{5.47}$$
$$+\frac{p_1}{t_1}[\theta^\infty_3(q_1-q_2)+p_2q_2(q_2-1)]$$
$$+\frac{1}{t_1-t_2}\left((q_1-q_2)p_1-\theta^\infty_2\right)\left((q_2-q_1)p_2-\theta^\infty_3\right),$$

$$H^{2+1+1+1}_{Gar,\ t_2}\begin{pmatrix}\theta^0+\theta^\infty_1,\theta^\infty_2 & t_1 & q_1,p_1\\ \theta^\infty_3,\theta^\infty_1-1 ; & t_2 ; & q_2,p_2\end{pmatrix} = H_{\mathrm{V}}\begin{pmatrix}\theta^1+\theta^\infty_3,\theta^0+\theta^1 & \\ -\theta^1 & ;t_2;q_2,p_2\end{pmatrix} \tag{5.48}$$
$$+\frac{p_2}{t_2}[\theta^\infty_2(q_2-q_1)+p_1q_1(q_1-1)]$$
$$+\frac{1}{t_2-t_1}\left((q_2-q_1)p_2-\theta^\infty_3\right)\left((q_1-q_2)p_1-\theta^\infty_2\right).$$

The gauge parameters satisfy

$$\frac{t_1q_1}{u}\frac{\partial u}{\partial t_1} = 2p_1q_1(q_1-1)-(2\theta^\infty_2+t_1)q_1-p_2q_2-\theta^1+\frac{p_2q_2(t_1q_1-t_2q_2)+\theta^\infty_3t_2q_2}{t_1-t_2}, \tag{5.49}$$

$$\frac{t_1q_2}{v}\frac{\partial v}{\partial t_1} = \frac{-p_1q_1(t_1q_1+(t_2-2t_1)q_2)+\theta^\infty_2t_1q_1}{t_1-t_2}-(p_1+\theta^\infty_2)q_2, \tag{5.50}$$

$$\frac{t_2q_1}{u}\frac{\partial u}{\partial t_2} = \frac{p_2q_2(t_2q_2+(t_1-2t_2)q_1)-\theta^\infty_3t_2q_2}{t_1-t_2}-(p_2+\theta^\infty_3)q_1, \tag{5.51}$$

$$\frac{t_2q_2}{v}\frac{\partial v}{\partial t_2} = 2p_2q_2(q_2-1)-(2\theta^\infty_3+t_2)q_2-p_1q_1-\theta^1-\frac{p_1q_1(t_2q_2-t_1q_1)+\theta^\infty_2t_1q_1}{t_1-t_2}. \tag{5.52}$$

Singularity pattern: 3+1

Spectral type: ((11))((1)),111

Riemann scheme is given as

$$\left(\begin{array}{ccc c}
\multicolumn{3}{c}{\overbrace{\quad x=0 \quad}} & x=\infty \\
0 & 0 & 0 & \theta^\infty_1\\
0 & 0 & \theta^0_1 & \theta^\infty_2\\
1 & -t & \theta^0_2 & \theta^\infty_3
\end{array}\right),$$

and then the Fuchs-Hukuhara relation is written as $\theta^0_1+\theta^0_2+\theta^\infty_1+\theta^\infty_2+\theta^\infty_3=0$.

The system of deformation equations is expressed as

$$
\left\{
\begin{aligned}
\frac{\partial Y}{\partial x} &= \left(\frac{A_0^{(-2)}}{x^3}+\frac{A_0^{(-1)}}{x^2}+\frac{A_0^{(0)}}{x}\right)Y,\\
\frac{\partial Y}{\partial t} &= \frac{A_0^{(-2)}}{x}Y,
\end{aligned}
\right.
\tag{5.53}
$$

where

$$
A_\xi^{(k)}=U^{-1}P^{-1}\hat{A}_\xi^{(k)}PU,\quad P=\begin{pmatrix} 1 & 0 & 0\\ \frac{a}{\theta^\infty_1-\theta^\infty_2} & 1 & 0\\ \frac{1}{\theta^\infty_1-\theta^\infty_3}\left(b+\frac{ac}{\theta^\infty_1-\theta^\infty_2}\right) & \frac{c}{\theta^\infty_2-\theta^\infty_3} & 1\end{pmatrix},\quad U=\mathrm{diag}(1,u,v),
$$

$$
\hat{A}_0^{(-2)}=\begin{pmatrix}1\\0\\0\end{pmatrix}(1,1,1),\quad \hat{A}_0^{(-1)}=\begin{pmatrix} p_1+p_2-t & q_2 & q_1\\ -p_2 & -p_2 & -p_2\\ -p_1 & -p_1 & -p_1\end{pmatrix},\quad \hat{A}_0^{(0)}=\begin{pmatrix}-\theta^\infty_1 & 0 & 0\\ -a & -\theta^\infty_2 & 0\\ -b & -c & -\theta^\infty_3\end{pmatrix},
$$

$$
a=p_2(p_2-q_2-t)+p_1p_2+\theta^0_1+\theta^\infty_2,\qquad b=p_1(p_1-q_1-t)+p_1p_2+\theta^\infty_3,\qquad c=p_1(q_2-q_1)+\theta^\infty_3.
$$

The Hamiltonian is given by

$$
\begin{aligned}
&H^{A_4}_{NY}\left({\theta^\infty_1-\theta^\infty_3-1,\theta^\infty_3 \atop -\theta^\infty_2,\theta^0_1+\theta^\infty_2};t;{q_1,p_1 \atop q_2,p_2}\right)\\
&=H_{\mathrm{IV}}\left(\theta^\infty_3,\theta^\infty_1-\theta^\infty_3-1;t;q_1,p_1\right)+H_{\mathrm{IV}}\left(\theta^0_1+\theta^\infty_2,\theta^\infty_1-\theta^\infty_2-\theta^\infty_3-1;t;q_2,p_2\right)+2p_1q_1p_2.
\end{aligned}
\tag{5.54}
$$

The gauge parameters satisfy

$$
\frac{1}{u}\frac{du}{dt}=-p_1-2p_2+q_2+t,\quad \frac{1}{v}\frac{dv}{dt}=q_1-2p_1-2p_2+t.
\tag{5.55}
$$

Spectral type: ((1)(1))((1)),21

Riemann scheme is given by

$$
\begin{pmatrix}
x=0 & \overbrace{\quad x=\infty \quad}^{} & & \\
0 & 0 & 0 & \theta^\infty_1\\
0 & -1 & t_1 & \theta^\infty_2\\
\theta^0 & -1 & t_2 & \theta^\infty_3
\end{pmatrix},
$$

and the Fuchs-Hukuhara relation is written as $\theta^0+\theta^\infty_1+\theta^\infty_2+\theta^\infty_3=0$.

The system of deformation equations is expressed as

$$
\left\{
\begin{aligned}
\frac{\partial Y}{\partial x} &= \left(A_\infty^{(-2)}x+A_\infty^{(-1)}+\frac{A_0^{(0)}}{x}\right)Y,\\
\frac{\partial Y}{\partial t_1} &= (-E_2x+B_1)Y,\qquad \frac{\partial Y}{\partial t_2}=(-E_3x+B_2)Y,
\end{aligned}
\right.
\tag{5.56}
$$

where

$$A_\xi^{(k)} = U^{-1}\hat{A}_\xi^{(k)}U, \quad B_i = U^{-1}\hat{B}_iU, \quad U = \mathrm{diag}(1,u,v),$$

$$\hat{A}_\infty^{(-2)} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \hat{A}_\infty^{(-1)} = \begin{pmatrix} 0 & -1 & -1 \\ -p_1q_1+\theta_2^\infty & -t_1 & 0 \\ -p_2q_2+\theta_3^\infty & 0 & -t_2 \end{pmatrix},$$

$$\hat{A}_0^{(0)} = \begin{pmatrix} 1 \\ p_1 \\ p_2 \end{pmatrix} (p_1q_1+p_2q_2+\theta^0,\ -q_1,\ -q_2), \quad E_2 = \mathrm{diag}(0,1,0), \quad E_3 = \mathrm{diag}(0,0,1),$$

$$\hat{B}_1 = \begin{pmatrix} 0 & 1 & 0 \\ p_1q_1-\theta_2^\infty & 0 & \frac{p_1(q_1-q_2)-\theta_2^\infty}{t_1-t_2} \\ 0 & \frac{p_2(q_2-q_1)-\theta_3^\infty}{t_1-t_2} & 0 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{p_1(q_1-q_2)-\theta_2^\infty}{t_2-t_1} \\ p_2q_2-\theta_3^\infty & \frac{p_2(q_2-q_1)-\theta_3^\infty}{t_2-t_1} & 0 \end{pmatrix}.$$

The Hamiltonians are given by

$$\begin{aligned}(5.57)\qquad & H_{Gar,t_1}^{3+1+1}\begin{pmatrix} \theta_2^\infty,\theta_3^\infty & ; & t_1 & ; & q_1,p_1 \\ \theta^0 & & t_2 & & q_2,p_2 \end{pmatrix} \\ &= H_{\mathrm{IV}}\left(\theta_2^\infty,\theta^0;t_1;q_1,p_1\right) + p_2q_2p_1 + \frac{1}{t_1-t_2}\{p_1(q_1-q_2)-\theta_2^\infty\}\{p_2(q_2-q_1)-\theta_3^\infty\},\end{aligned}$$

$$\begin{aligned}(5.58)\qquad & H_{Gar,t_2}^{3+1+1}\begin{pmatrix} \theta_2^\infty,\theta_3^\infty & ; & t_1 & ; & q_1,p_1 \\ \theta^0 & & t_2 & & q_2,p_2 \end{pmatrix} \\ &= H_{\mathrm{IV}}\left(\theta_3^\infty,\theta^0;t_2;q_2,p_2\right) + p_1q_1p_2 + \frac{1}{t_2-t_1}\{p_1(q_1-q_2)-\theta_2^\infty\}\{p_2(q_2-q_1)-\theta_3^\infty\}.\end{aligned}$$

The gauge parameters satisfy

$$(5.59)\qquad (t_1-t_2)\frac{1}{u}\frac{\partial u}{\partial t_1} = p_2(q_1-q_2)+(t_1-t_2)(q_1+t_1)+\theta_3^\infty, \quad (t_1-t_2)\frac{1}{v}\frac{\partial v}{\partial t_1} = p_1(q_2-q_1)+\theta_2^\infty,$$

$$(5.60)\qquad (t_2-t_1)\frac{1}{u}\frac{\partial u}{\partial t_2} = p_2(q_1-q_2)+\theta_3^\infty, \quad (t_2-t_1)\frac{1}{v}\frac{\partial v}{\partial t_2} = p_1(q_2-q_1)+(t_2-t_1)(q_2+t_2)+\theta_2^\infty.$$

Singularity pattern: 2+2

Spectral type: (11)(1),(11)(1)

Riemann scheme is given as

$$\left(\begin{array}{cc} \overbrace{\begin{matrix} x=0 & \\ \end{matrix}} & \overbrace{\begin{matrix} x=\infty & \\ \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & \theta_1^0 \\ t & \theta_2^0 \end{matrix} & \begin{matrix} 0 & \theta_1^\infty \\ 0 & \theta_2^\infty \\ 1 & \theta_3^\infty \end{matrix} \end{array}\right),$$

and the Fuchs-Hukuhara relation is written as $\theta_1^0+\theta_2^0+\theta_1^\infty+\theta_2^\infty+\theta_3^\infty=0$.

The system of deformation equations is expressed as

$$(5.61)\qquad \left\{\begin{aligned} \frac{\partial Y}{\partial x} &= \left(\frac{A_0^{(-1)}}{x^2}+\frac{A_0^{(0)}}{x}+A_\infty\right)Y, \\ \frac{\partial Y}{\partial t} &= -\frac{A_0^{(-1)}}{tx}Y, \end{aligned}\right.$$

where

$$A_\xi^{(k)} = U^{-1}P^{-1}\hat{A}_\xi^{(k)}PU, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ -p_1q_1 & 1 & 0 \\ -p_2q_2 & \frac{p_2q_2(q_1-q_2)+\theta_1^\infty q_2}{(\theta_2^\infty-\theta_1^\infty)q_1} & 1 \end{pmatrix}, \quad U = \mathrm{diag}(1,u,v),$$

$$\hat{A}_\infty = \begin{pmatrix} -1 \\ p_1q_1 \\ p_2q_2 \end{pmatrix}(1,0,0), \quad \hat{A}_0^{(0)} = \begin{pmatrix} -p_1q_1-p_2q_2-\theta_3^\infty & -1 & -1 \\ q_1(p_1q_1-\theta_1^0-\theta_2^\infty) & p_1q_1-\theta_2^\infty & p_1q_1 \\ q_2(p_2q_2-\theta_1^\infty) & q_2(p_2q_2-\theta_1^\infty)/q_1 & p_2q_2-\theta_1^\infty \end{pmatrix},$$

$$\hat{A}_0^{(-1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(t, t/q_1, t/q_2).$$

The Hamiltonian is given by

$$\begin{aligned} (5.62) \qquad & tH_{FS}^{A_3}\left(-\theta_1^0, -\theta_3^\infty, -\theta_2^\infty, -\theta_1^\infty; t; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix}\right) \\ &= tH_{\mathrm{III}}(D_6)\left(-\theta_1^0-\theta_2^\infty, \theta_3^\infty-\theta_2^\infty; t; q_1, p_1\right) + tH_{\mathrm{III}}(D_6)\left(-\theta_1^\infty, \theta_3^\infty-\theta_1^\infty; t; q_2, p_2\right) \\ &\quad + p_1q_2\{p_2(q_1+q_2)-\theta_1^\infty\}. \end{aligned}$$

The gauge parameters satisfy

$$(5.63) \qquad tq_1\frac{1}{u}\frac{du}{dt} = q_2(p_2q_2-\theta_1^\infty)+t, \qquad \frac{1}{v}\frac{dv}{dt} = \frac{1}{q_2}.$$

Spectral type: (2)(1),(1)(1)(1)

Riemann scheme is given by

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c}{\overbrace{\quad x=0 \quad}} & \multicolumn{2}{c}{\overbrace{\quad x=\infty \quad}} \\ 0 & 0 & 0 & \theta_1^\infty \\ 0 & 0 & -t_1 & \theta_2^\infty \\ 1 & \theta^0 & -t_2 & \theta_3^\infty \end{array} \right),$$

and the Fuchs-Hukuhara relation is written as $\theta^0+\theta_1^\infty+\theta_2^\infty+\theta_3^\infty = 0$.

The system of deformation equations is expressed as

$$(5.64) \qquad \begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0^{(-1)}}{x^2} + \dfrac{A_0^{(0)}}{x} + A_\infty\right)Y, \\ \dfrac{\partial Y}{\partial t_1} = (E_2x+B_1)Y, \qquad \dfrac{\partial Y}{\partial t_2} = (E_3x+B_2)Y, \end{cases}$$

where

$$A^{(k)}_\xi = U^{-1}\hat{A}^{(k)}_\xi U, \quad B_i = U^{-1}\hat{B}_i U, \quad U = \mathrm{diag}(1,u,v),$$

$$A_\infty = \begin{pmatrix} 0 & & \\ & t_1 & \\ & & t_2 \end{pmatrix}, \quad \hat{A}^{(-1)}_0 = \begin{pmatrix} 1 \\ p_1 \\ p_2 \end{pmatrix} (-p_1 - p_2 + 1,\ 1,\ 1),$$

$$\hat{A}^{(0)}_0 = \begin{pmatrix} -\theta^\infty_1 & -q_1 & -q_2 \\ -p_1q_1(p_1+p_2-1) + \theta^\infty_2(p_2-1) + (\theta^\infty_2 - \theta^\infty_1)p_1 & -\theta^\infty_2 & p_1(q_1-q_2) - \theta^\infty_2 \\ -p_2q_2(p_1+p_2-1) + \theta^\infty_3(p_1-1) + (\theta^\infty_3 - \theta^\infty_1)p_2 & p_2(q_2-q_1) - \theta^\infty_3 & -\theta^\infty_3 \end{pmatrix},$$

$$\hat{B}_1 = \begin{pmatrix} 0 & \frac{(\hat{A}^{(0)}_0)_{12}}{t_1} & 0 \\ \frac{(\hat{A}^{(0)}_0)_{21}}{t_1} & 0 & \frac{(\hat{A}^{(0)}_0)_{23}}{t_1-t_2} \\ 0 & \frac{(\hat{A}^{(0)}_0)_{32}}{t_1-t_2} & 0 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} 0 & 0 & \frac{(\hat{A}^{(0)}_0)_{13}}{t_2} \\ 0 & 0 & \frac{(\hat{A}^{(0)}_0)_{23}}{t_2-t_1} \\ \frac{(\hat{A}^{(0)}_0)_{31}}{t_2} & \frac{(\hat{A}^{(0)}_0)_{32}}{t_2-t_1} & 0 \end{pmatrix}.$$

The Hamiltonians are given by

(5.65)
$$t_1 H^{\frac{3}{2}+1+1+1}_{Gar,t_1}\begin{pmatrix} \theta^\infty_2, \theta^\infty_3 & t_1 & q_1, p_1 \\ \theta^\infty_1 & ; t_2 ; & q_2, p_2 \end{pmatrix} = t_1 H_{\mathrm{III}}(D_6)\left(-\theta^\infty_2, \theta^\infty_1 - \theta^\infty_2; t_1; q_1, p_1\right)$$
$$+ q_1(q_1p_1p_2 - \theta^\infty_2 p_2) + \frac{t_1}{t_1-t_2}(p_1(q_1-q_2) - \theta^\infty_2)(p_2(q_2-q_1) - \theta^\infty_3),$$

(5.66)
$$t_2 H^{\frac{3}{2}+1+1+1}_{Gar,t_2}\begin{pmatrix} \theta^\infty_2, \theta^\infty_3 & t_1 & q_1, p_1 \\ \theta^\infty_1 & ; t_2 ; & q_2, p_2 \end{pmatrix} = t_2 H_{\mathrm{III}}(D_6)\left(-\theta^\infty_3, \theta^\infty_1 - \theta^\infty_3; t_2; q_2, p_2\right)$$
$$+ q_2(q_2p_1p_2 - \theta^\infty_3 p_1) + \frac{t_2}{t_2-t_1}(p_1(q_1-q_2) - \theta^\infty_2)(p_2(q_2-q_1) - \theta^\infty_3).$$

The gauge parameters satisfy

$$t_1(t_1-t_2)\frac{1}{u}\frac{\partial u}{\partial t_1} = (t_1-t_2)(1-2p_1)q_1 + p_2(t_2q_1 - t_1q_2) + \theta^\infty_3 t_1, \tag{5.67}$$

$$t_1(t_1-t_2)\frac{1}{v}\frac{\partial v}{\partial t_1} = t_1(\theta^\infty_2 - 2p_1q_1) + p_1(t_2q_1 + t_1q_2), \tag{5.68}$$

$$t_2(t_2-t_1)\frac{1}{u}\frac{\partial u}{\partial t_2} = t_2(\theta^\infty_3 - 2p_2q_2) + p_2(t_2q_1 + t_1q_2), \tag{5.69}$$

$$t_2(t_2-t_1)\frac{1}{v}\frac{\partial v}{\partial t_2} = (t_2-t_1)(1-2p_2)q_2 + p_1(t_1q_2 - t_2q_1) + \theta^\infty_2 t_2. \tag{5.70}$$

Singularity pattern: 4

Spectral type: (((1)(1)))(((1)))

Riemann scheme is given by

$$\left(\begin{array}{cccc} \multicolumn{4}{c}{\overbrace{\qquad x=\infty \qquad}} \\ 0 & 0 & 0 & \theta^\infty_1 \\ 1 & 0 & t_1 & \theta^\infty_2 \\ 1 & 0 & t_2 & \theta^\infty_3 \end{array} \right),$$

and the Fuchs-Hukuhara relation is written as $\theta^\infty_1 + \theta^\infty_2 + \theta^\infty_3 = 0$.

The system of deformation equation is expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(A^{(-3)}_\infty x^2 + A^{(-2)}_\infty x + A^{(-1)}_\infty\right) Y, \\ \dfrac{\partial Y}{\partial t_1} = (-E_2 x + B_1)Y, \qquad \dfrac{\partial Y}{\partial t_2} = (-E_3 x + B_2)Y, \end{cases} \tag{5.71}$$

where

$$A_\infty^{(k)} = U^{-1}\hat{A}_\infty^{(k)}U, \quad B_i = U^{-1}\hat{B}_iU, \quad U = \mathrm{diag}(1,u,v), \quad (i=1,2),$$

$$\hat{A}_\infty^{(-3)} = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad \hat{A}_\infty^{(-2)} = \begin{pmatrix} 0 & -1 & -1 \\ -p_1 & 0 & 0 \\ -p_2 & 0 & 0 \end{pmatrix}, \quad \hat{A}_\infty^{(-1)} = \begin{pmatrix} -p_1-p_2 & -q_1 & -q_2 \\ p_1q_1-\theta_2^\infty & p_1-t_1 & p_1 \\ p_2q_2-\theta_3^\infty & p_2 & p_2-t_2 \end{pmatrix}.$$

$$\hat{B}_1 = \begin{pmatrix} 0 & -1 & 0 \\ -p_1 & \frac{1}{t_1-t_2}\{p_2(q_1-q_2)+\theta_3^\infty\}+q_1 & \frac{1}{t_1-t_2}\{p_1(q_1-q_2)-\theta_2^\infty\} \\ 0 & \frac{1}{t_1-t_2}\{p_2(q_2-q_1)-\theta_3^\infty\} & \frac{1}{t_1-t_2}\{p_1(q_2-q_1)+\theta_2^\infty\} \end{pmatrix},$$

$$\hat{B}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{t_2-t_1}\{p_2(q_1-q_2)+\theta_3^\infty\} & \frac{1}{t_2-t_1}\{p_1(q_1-q_2)-\theta_2^\infty\} \\ -p_2 & \frac{1}{t_2-t_1}\{p_2(q_2-q_1)-\theta_3^\infty\} & \frac{1}{t_2-t_1}\{p_1(q_2-q_1)+\theta_2^\infty\}+q_2 \end{pmatrix},$$

$$E_2 = \mathrm{diag}(0,1,0), \qquad E_3 = \mathrm{diag}(0,0,1).$$

The Hamiltonians are given by

$$\begin{aligned}(5.72)\qquad & H_{Gar,t_1}^{\frac{5}{2}+1+1}\left(\theta_2^\infty,\theta_3^\infty;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right) \\ &= H_{\mathrm{II}}\left(-\theta_2^\infty;t_1;q_1,p_1\right)+p_1p_2+\frac{1}{t_1-t_2}(p_1(q_1-q_2)-\theta_2^\infty)(p_2(q_2-q_1)-\theta_3^\infty),\end{aligned}$$

$$\begin{aligned}(5.73)\qquad & H_{Gar,t_1}^{\frac{5}{2}+1+1}\left(\theta_2^\infty,\theta_3^\infty;\begin{matrix}t_1\\t_2\end{matrix};\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right) \\ &= H_{\mathrm{II}}\left(-\theta_3^\infty;t_2;q_2,p_2\right)+p_1p_2+\frac{1}{t_2-t_1}(p_1(q_1-q_2)-\theta_2^\infty)(p_2(q_2-q_1)-\theta_3^\infty).\end{aligned}$$

The gauge parameters satisfy

$$\frac{\partial u}{\partial t_1}=0, \quad \frac{\partial v}{\partial t_1}=0, \quad \frac{\partial u}{\partial t_2}=0, \quad \frac{\partial v}{\partial t_2}=0. \tag{5.74}$$

5.3 Sasano system

Singularity pattern: 1+1+1+1

Spectral type: 31,22,22,1111

Riemann scheme is given by

$$\begin{pmatrix} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ 0 & 0 & 0 & \theta_2^\infty \\ 0 & \theta^1 & \theta^t & \theta_3^\infty \\ \theta^0 & \theta^1 & \theta^t & \theta_4^\infty \end{pmatrix},$$

and the Fuchs relation is written as $\theta^0+2\theta^1+2\theta^t+\theta_1^\infty+\theta_2^\infty+\theta_3^\infty+\theta_4^\infty=0$.

The system of deformation equation is expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0}{x}+\dfrac{A_1}{x-1}+\dfrac{A_t}{x-t}\right)Y, \\ \dfrac{\partial Y}{\partial t} = -\dfrac{A_t}{x-t}Y. \end{cases} \tag{5.75}$$

Here A_0, A_1, and A_t is given as follows:

$$A_\xi = U^{-1}P^{-1}\hat{A}_\xi PU, \quad (\xi = 0, 1, t), \quad U = \mathrm{diag}(1, u, v, w)$$

$$\hat{A}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left(\theta^0,\ (\hat{A}_0)_{12},\ -f_1 + \frac{q_1 f_1 + q_2 - q_1}{t},\ -1 + \frac{q_1}{t}\right),$$

$$\hat{A}_\xi = \begin{pmatrix} I_2 \\ \hat{B}_\xi \end{pmatrix} (\theta^\xi I_2 - \hat{C}_\xi \hat{B}_\xi,\ \hat{C}_\xi), \quad (\xi = 1, t), \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\hat{a}_2^\infty}{\theta_1^\infty - \theta_2^\infty} & 1 & 0 & 0 \\ \frac{\hat{a}_3^\infty}{\theta_1^\infty - \theta_3^\infty} & 0 & 1 & 0 \\ \frac{\hat{a}_4^\infty}{\theta_1^\infty - \theta_4^\infty} & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{B}_1 = \begin{pmatrix} p_2 q_2 - \theta_3^\infty & -p_2(q_2 - q_1) + f_2 + \theta_3^\infty \\ (1 - f_1)(p_2 q_2 - \theta_4^\infty) + p_1 q_2 & -(q_2 - q_1)(p_1 + p_2(1 - f_1)) - \theta_4^\infty f_1 + f_3 \end{pmatrix},$$

$$\hat{C}_1 = \begin{pmatrix} f_1 & 1 \\ f_1 - 1 & 1 \end{pmatrix}, \quad \hat{B}_t = \begin{pmatrix} tp_2 & f_2 \\ t(p_1 + p_2(1 - f_1)) & f_3 \end{pmatrix}, \quad \hat{C}_t = \begin{pmatrix} \frac{q_1(1-f_1) - q_2}{t} & -\frac{q_1}{t} \\ 1 - f_1 & -1 \end{pmatrix},$$

$$\begin{aligned}
(\theta_3^\infty - \theta_4^\infty) f_1 &= p_1(q_2 - q_1) + \theta^1 + \theta^t + \theta_2^\infty + \theta_3^\infty, \\
(\theta_3^\infty - \theta_2^\infty) f_2 &= (p_2(q_2 - q_1) - \theta_3^\infty)(p_2(q_1(1 - f_1) - q_2) + \theta^1 + \theta_3^\infty) \\
&\qquad - p_2 q_1((p_1 + p_2(1 - f_1))(q_2 - q_1) + \theta_4^\infty f_1), \\
(\theta_4^\infty - \theta_2^\infty) f_3 &= (p_1 + p_2(1 - f_1))\{(\theta_3^\infty - \theta_4^\infty) q_1 f_1 \\
&\qquad + (q_1 - q_2)(q_1 p_1 + q_2 p_2 - \theta^1 - \theta_3^\infty - \theta_4^\infty)\} + \theta_4^\infty(\theta^1 + \theta_4^\infty) f_1, \\
(\hat{A}_0)_{12} &= -(p_1 + p_2)(q_2 - q_1) + (\theta_3^\infty - \theta_4^\infty) f_1 + f_2\left(f_1 - \frac{q_1 f_1 + q_2 - q_1}{t}\right) + f_3\left(1 - \frac{q_1}{t}\right),
\end{aligned}$$

$$\begin{aligned}
\hat{a}_2^\infty &= (\hat{C}_1 \hat{B}_1 + \hat{C}_t \hat{B}_t - (\theta^1 + \theta^t) I_2)_{21}, \quad \hat{a}_3^\infty = (\hat{B}_1(\hat{C}_1 \hat{B}_1 - \theta^1 I_2) + \hat{B}_t(\hat{C}_t \hat{B}_t - \theta^t I_2))_{11}, \\
\hat{a}_4^\infty &= (\hat{B}_1(\hat{C}_1 \hat{B}_1 - \theta^1 I_2) + \hat{B}_t(\hat{C}_t \hat{B}_t - \theta^t I_2))_{21}.
\end{aligned}$$

The Hamiltonian is given by

$$\begin{aligned}
&t(t-1) H_{Ss}^{D_6}\left({\theta_1^\infty - \theta_2^\infty, \theta_2^\infty - \theta_3^\infty, \theta_3^\infty - \theta_4^\infty \atop \theta_4^\infty, \theta^1, \theta^t}; t; {q_1, p_1 \atop q_2, p_2}\right) \\
&= t(t-1) H_{\mathrm{VI}}\left({-\theta^0 - \theta^1 - \theta^t - \theta_1^\infty - \theta_3^\infty, -\theta^t - \theta_2^\infty + \theta_3^\infty \atop -\theta^1 - \theta_2^\infty - \theta_3^\infty, \theta^0 + \theta^1 + \theta^t + \theta_2^\infty + \theta_3^\infty + 1}; t; q_1, p_1\right) \\
&\quad + t(t-1) H_{\mathrm{VI}}\left({\theta_3^\infty, \theta^1 \atop \theta^t, -\theta_1^\infty + \theta_2^\infty - \theta_3^\infty + \theta_4^\infty + 1}; t; q_2, p_2\right) \\
&\quad + 2(q_1 - 1) p_2 q_2 (p_1(q_1 - t) + \theta^0 + \theta^1 + \theta^t + \theta_1^\infty + \theta_3^\infty).
\end{aligned} \tag{5.76}$$

The gauge parameters satisfy

$$\frac{1}{u}\frac{du}{dt} = \frac{1}{\hat{a}_2^\infty}\left(\frac{d\hat{a}_2^\infty}{dt} + (\theta_1^\infty - \theta_2^\infty) p_1\right), \tag{5.77}$$

$$\frac{1}{v}\frac{dv}{dt} = \frac{1}{\hat{a}_3^\infty}\left(\frac{d\hat{a}_3^\infty}{dt} + (\theta_1^\infty - \theta_3^\infty)(f_2 p_1 + p_2(p_1 q_1 + p_2 q_2 + \theta^t))\right), \tag{5.78}$$

$$\frac{1}{w}\frac{dw}{dt} = \frac{1}{\hat{a}_4^\infty}\left(\frac{d\hat{a}_4^\infty}{dt} + (\theta_1^\infty - \theta_4^\infty)(f_3 p_1 + (p_1 + p_2(1 - f_1))(p_1 q_1 + p_2 q_2 + \theta^t))\right). \tag{5.79}$$

Singularity pattern: 2+1+1

Spectral type: $(2)(2), 31, 1111$

Riemann scheme is given by

$$\left(\begin{array}{cccc} x=0 & \overbrace{x=1}^{} & & x=\infty \\ 0 & 0 & 0 & \theta^\infty_1 \\ 0 & 0 & 0 & \theta^\infty_2 \\ 0 & t & \theta^1 & \theta^\infty_3 \\ \theta^0 & t & \theta^1 & \theta^\infty_4 \end{array}\right),$$

and the Fuchs-Hukuhara relation is written as $\theta^0 + 2\theta^1 + \theta^\infty_1 + \theta^\infty_2 + \theta^\infty_3 + \theta^\infty_4 = 0$.

The system of deformation equation is expressed as

$$(5.80)\qquad \begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_1^{(-1)}}{(x-1)^2} + \dfrac{A_1^{(0)}}{x-1} + \dfrac{A_0^{(0)}}{x}\right)Y, \\ \dfrac{\partial Y}{\partial t} = -\dfrac{1}{x-1}\left(\dfrac{A_1^{(-1)}}{t}\right)Y, \end{cases}$$

where

$$A_\xi^{(k)} = U^{-1}P^{-1}\hat{A_\xi}^{(k)}PU, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{a_{21}}{\theta^\infty_1 - \theta^\infty_2} & 1 & 0 & 0 \\ -\frac{a_{31}}{\theta^\infty_1 - \theta^\infty_3} & 0 & 1 & 0 \\ -\frac{a_{41}}{\theta^\infty_1 - \theta^\infty_4} & 0 & 0 & 1 \end{pmatrix}, \quad U = \mathrm{diag}(1, u, v, w),$$

$$\hat{A_1}^{(-1)} = \begin{pmatrix} I_2 \\ \hat{B}_1 \end{pmatrix}\begin{pmatrix} tI_2 + \hat{C}_1\hat{B}_1 & -\hat{C}_1 \end{pmatrix}, \quad \hat{C}_1 = \begin{pmatrix} f_1 & 1 \\ f_1 - 1 & 1 \end{pmatrix}, \ \hat{B}_1 = \begin{pmatrix} p_2 & f_2 \\ p_1 + (1-f_1)p_2 & f_3 \end{pmatrix},$$

$$\hat{A_1}^{(0)} = \begin{pmatrix} -\theta^0 - \theta^\infty_1 & a_{12} & f_1(1-q_1) + q_1 - q_2 & 1 - q_1 \\ \theta^1 + \theta^\infty_2 + \theta^\infty_4 - p_1(q_1 - 1) & -\theta^\infty_2 & 0 & 0 \\ a_{31} & 0 & -\theta^\infty_3 & 0 \\ a_{41} & 0 & 0 & -\theta^\infty_4 \end{pmatrix},$$

$$\hat{A_0}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} \theta^0 & f_4 & f_1(q_1-1) + q_2 - q_1 & q_1 - 1 \end{pmatrix},$$

$$\begin{aligned} a_{12} &= (q_1 - 1)(f_1 f_2 + f_3) - (q_1 - q_2)(f_2 + p_2) - (\theta^1 + \theta^\infty_2 + \theta^\infty_3), \\ a_{31} &= (p_2 + f_2)p_1(1 - q_1) + p_2((t + p_2)(1 - q_2) - \theta^0 - \theta^\infty_1 + \theta^\infty_3) + t\theta^\infty_3 \\ &\quad + f_2(\theta^1 + \theta^\infty_2 + \theta^\infty_4), \\ a_{41} &= (p_1 - (f_1 - 1)p_2)((p_2 + t)(1 - q_2) + p_1(1 - q_1) - \theta^0 - \theta^\infty_1 + \theta^\infty_4) \\ &\quad - f_3(p_1(q_2 - 1) - (f_1 - 1)(\theta^\infty_3 - \theta^\infty_4)) - (f_1 - 1)t\theta^\infty_4, \end{aligned}$$

$$f_1 = \frac{p_1(q_2 - q_1) + \theta^1 + \theta^\infty_2 + \theta^\infty_3}{\theta^\infty_3 - \theta^\infty_4}, \quad f_2 = \frac{p_2((p_2 + t)(q_1 - q_2) + \theta^1 + \theta^\infty_2 + \theta^\infty_3) + t\theta^\infty_3}{\theta^\infty_3 - \theta^\infty_2},$$

$$f_3 = \frac{(p_1 + p_2(1 - f_1))((p_2 + t)(q_1 - q_2) + \theta^1 + \theta^\infty_2 + \theta^\infty_3) - t\theta^\infty_4 f_1}{\theta^\infty_4 - \theta^\infty_2},$$

$$f_4 = (p_2 + f_2)(q_1 - q_2) + (f_1 f_2 + f_3)(1 - q_1) + (\theta^1 + \theta^\infty_2 + \theta^\infty_3).$$

The Hamiltonian is given by

$$
\begin{aligned}
(5.81)\qquad & tH^{D_5}_{Ss}\begin{pmatrix}\theta^0+\theta^1+\theta^\infty_2+\theta^\infty_3, -\theta^\infty_1+\theta^\infty_4+1, \theta^\infty_1-\theta^\infty_2-\theta^\infty_3 & ;t; & q_1,p_1\\ -\theta^0-\theta^1-\theta^\infty_2-\theta^\infty_4, -\theta^0-\theta^1-\theta^\infty_3-\theta^\infty_4-1 & & q_2,p_2\end{pmatrix}\\
&= tH_{\mathrm{V}}\begin{pmatrix}-\theta^0-\theta^1-\theta^\infty_3-\theta^\infty_4-1, \theta^0+\theta^1+\theta^\infty_1+\theta^\infty_2+\theta^\infty_3-\theta^\infty_4-1 & ;t;q_1,p_1\\ -\theta^\infty_1+\theta^\infty_4+1 & \end{pmatrix}\\
&+ tH_{\mathrm{V}}\begin{pmatrix}\theta^0+\theta^1+2\theta^\infty_1+\theta^\infty_3-1, \theta^0+\theta^1+\theta^\infty_1+\theta^\infty_2-\theta^\infty_3+\theta^\infty_4-1 & ;t;q_2,p_2\\ -\theta^0-\theta^1-2\theta^\infty_1+1 & \end{pmatrix}\\
&+2p_2q_1(p_1(q_1-1)+\theta^0+\theta^1+\theta^\infty_1+\theta^\infty_3).
\end{aligned}
$$

The gauge parameters satisfy

$$
\begin{aligned}
(5.82)\qquad & -\frac{t}{u}\frac{du}{dt}=(t+2p_2)(1-q_1)+p_1,\quad -\frac{t}{v}\frac{dv}{dt}=(t+2p_2)(1-q_2)+p_1+\theta^1+2\theta^\infty_3,\\
& -\frac{t}{w}\frac{dw}{dt}=(t+2p_1+2p_2)(1-q_1)-\theta^1-2\theta^\infty_4.
\end{aligned}
$$

Spectral type: $(11)(11),31,22$

Riemann scheme is given by

$$
\begin{pmatrix}
x=0 & x=1 & \overbrace{x=\infty}^{} & \\
0 & 0 & 0 & \theta^\infty_1\\
0 & 0 & 0 & \theta^\infty_2\\
0 & \theta^1 & t & \theta^\infty_3\\
\theta^0 & \theta^1 & t & \theta^\infty_4
\end{pmatrix},
$$

and the Fuchs-Hukuhara relation is written as $\theta^0+2\theta^1+\theta^\infty_1+\theta^\infty_2+\theta^\infty_3+\theta^\infty_4=0$.

The system of deformation equation is expressed as

$$
(5.83)\qquad \begin{cases}\dfrac{\partial Y}{\partial x}=\left(\dfrac{A_0}{x}+\dfrac{A_1}{x-1}+A_\infty\right)Y,\\[2mm] \dfrac{\partial Y}{\partial t}=(-E_2\otimes I_2x+B_1)Y,\end{cases}
$$

where

$$
A_\xi=U^{-1}\hat{A}_\xi U,\ B_1=U^{-1}\hat{B}_1U,\ U=\mathrm{diag}(1,u,v,w),
$$

$$
\begin{aligned}
\hat{A}_0&=\begin{pmatrix}-f_1-p_1q_1\\ -f_2-p_2q_2+f_5(f_1+p_1q_1)\\ 1\\ \frac{f_2}{\theta^\infty_4-\theta^\infty_3}\end{pmatrix}\times\\
&\left(\frac{-f_2+p_2(q_2-1)q_1-\theta^1-\theta^\infty_1-\theta^\infty_4}{(\theta^\infty_1-\theta^\infty_2)q_1}+1 \quad 1-\frac{1}{q_2} \quad f_1-\theta^1-\theta^\infty_3+\frac{f_2(f_3+\theta^1+\theta^\infty_3)}{\theta^\infty_3-\theta^\infty_4} \quad f_3+\theta^1+\theta^\infty_4\right),\\
\hat{A}_1&=\begin{pmatrix}-f_1 & -f_3\\ \frac{1}{\theta^\infty_1-\theta^\infty_2}\left\{(1-\frac{1}{q_1})f_4+\theta^\infty_2f_2\right\} & \frac{1}{\theta^\infty_1-\theta^\infty_2}\left\{(\frac{1}{q_1}-\frac{1}{q_2})f_4+(\theta^1+\theta^\infty_1+\theta^\infty_4)\theta^\infty_2\right\}\\ 1 & 0\\ \frac{f_2}{\theta^\infty_4-\theta^\infty_3} & 1\end{pmatrix}\times\\
&\begin{pmatrix}-f_5-1 & -1 & \frac{f_2(f_3+\theta^1+\theta^\infty_3)}{\theta^\infty_4-\theta^\infty_3}-f_1+\theta^1 & -f_3-\theta^1-\theta^\infty_4\\ -f_5 & -1 & \frac{\theta^\infty_3f_2}{\theta^\infty_4-\theta^\infty_3} & -\theta^\infty_4\end{pmatrix},\\
A_\infty&=\begin{pmatrix}O_2 & O_2\\ O_2 & -tI_2\end{pmatrix},\quad \hat{B}_1=\begin{pmatrix}O & \frac{1}{t}[A_0+A_1]_{1,2}\\ \frac{1}{t}[A_0+A_1]_{2,1} & O\end{pmatrix},
\end{aligned}
$$

$$f_1 = p_1q_1(q_1-1) - \theta^\infty_1 q_1, \quad f_2 = p_2q_2(q_2-1) - (\theta^1+\theta^\infty_2+\theta^\infty_4)q_2, \quad f_3 = \frac{q_1}{q_2}(p_1(q_2-q_1)+\theta^\infty_1,$$

$$f_4 = p_2q_2f_1 - p_1q_1f_2, \quad f_5 = \frac{1}{(\theta^\infty_1-\theta^\infty_2)q_1}(p_2q_2(q_1-q_2)+(\theta^1+\theta^\infty_2+\theta^\infty_4)q_2).$$

The Hamiltonian is given by

$$(5.84) \quad tH^{A_5}_{NY}\begin{pmatrix} \theta^1, \theta^\infty_1, -\theta^1-\theta^\infty_1-\theta^\infty_4 & ;t; & q_1,p_1 \\ \theta^1+\theta^\infty_2+\theta^\infty_4, -\theta^1-\theta^\infty_2-\theta^\infty_3 & & q_2,p_2 \end{pmatrix}$$
$$= tH_{\mathrm{V}}\begin{pmatrix} \theta^1+\theta^\infty_1, \theta^0+\theta^1 & ;t;q_1,p_1 \\ -\theta^1 & \end{pmatrix} + tH_{\mathrm{V}}\begin{pmatrix} \theta^1-\theta^\infty_1+\theta^\infty_2, \theta^0+\theta^1 & ;t;q_2,p_2 \\ \theta^\infty_1+\theta^\infty_4 & \end{pmatrix} + 2p_1p_2q_1(q_2-1).$$

The gauge parameters satisfy

$$(5.85) \quad -tq_1q_2\frac{1}{u}\frac{du}{dt} = q_1q_2(p_2-2p_1+\theta^\infty_2-\theta^\infty_1) + q_1(2p_1q_1-\theta^1-\theta^4) - \theta^1q_2,$$

$$(5.86) \quad tq_1\frac{1}{v}\frac{dv}{dt} = q_1p_2(1-2q_2) - 2p_1q_1(q_1-1) + (t+\theta^1+2\theta^\infty_1+\theta^\infty_2+\theta^\infty_4)q_1 + \theta^1,$$

$$(5.87) \quad tq_1q_2\frac{1}{w}\frac{dw}{dt} = q_1q_2(2p_1+t+\theta^\infty_1-\theta^\infty_4) - q_1(2p_1q_1-\theta^\infty_1-\theta^\infty_4) + \theta^1q_2.$$

<u>Spectral type: $(111)(1), 22, 22$</u>

Riemann scheme is given by

$$\begin{pmatrix} x=0 & x=1 & \overbrace{x=\infty}^{} & \\ 0 & 0 & t & \theta^\infty_1 \\ 0 & 0 & 0 & \theta^\infty_2 \\ \theta^0 & \theta^1 & 0 & \theta^\infty_3 \\ \theta^0 & \theta^1 & 0 & \theta^\infty_4 \end{pmatrix},$$

and the Fuchs-Hukuhara relation is written as $2\theta^0+2\theta^1+\theta^\infty_1+\theta^\infty_2+\theta^\infty_3+\theta^\infty_4=0$.

The system of deformation equation is expressed as

$$(5.88) \quad \begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0}{x} + \dfrac{A_1}{x-1} + A_\infty\right)Y, \\ \dfrac{\partial Y}{\partial t} = (-E_1x+B)Y. \end{cases}$$

Here

$$A_\xi = U^{-1}\hat{A}_\xi U, \quad B = U^{-1}\hat{B}U, \quad U = \mathrm{diag}(1,u,v,w),$$
$$\hat{A}_\xi = \begin{pmatrix} I_2 \\ \hat{B}_\xi \end{pmatrix}(\theta^\xi I_2 - \hat{C}_\xi\hat{B}_\xi,\ \hat{C}_\xi) \quad (\xi=0,1), \quad A_\infty = -tE_1, \quad E_1 = \mathrm{diag}(1,0,0,0),$$

$$\hat{B}_0 = \begin{pmatrix} \mu_2 & f_2 \\ \mu_1-(f_1-1)\mu_2 & f_3 \end{pmatrix}, \quad \hat{C}_0 = \begin{pmatrix} (1-f_1)\lambda_1-\lambda_2 & -\lambda_1 \\ 1-f_1 & -1 \end{pmatrix},$$
$$\hat{B}_1 = \begin{pmatrix} \mu_2\lambda_2-\theta^\infty_3 & f_2+\mu_2(\lambda_1-\lambda_2)+\theta^\infty_3 \\ \mu_1\lambda_2+(1-f_1)(\mu_2\lambda_2-\theta^\infty_4) & f_3+(\lambda_1-\lambda_2)(\mu_2(1-f_1)+\mu_1)-f_1\theta^\infty_4 \end{pmatrix}, \hat{C}_1 = \begin{pmatrix} f_1 & 1 \\ f_1-1 & 1 \end{pmatrix},$$
$$\hat{B} = \frac{1}{t}\begin{pmatrix} 0 & (\hat{A}_0+\hat{A}_1)_{12} & (\hat{A}_0+\hat{A}_1)_{13} & (\hat{A}_0+\hat{A}_1)_{14} \\ (\hat{A}_0+\hat{A}_1)_{21} & 0 & 0 & 0 \\ (\hat{A}_0+\hat{A}_1)_{31} & 0 & 0 & 0 \\ (\hat{A}_0+\hat{A}_1)_{41} & 0 & 0 & 0 \end{pmatrix},$$

$$f_1=\frac{\mu_1(\lambda_2-\lambda_1)+\theta^0+\theta^1+\theta^\infty_2+\theta^\infty_3}{\theta^\infty_3-\theta^\infty_4},$$
$$f_2=\frac{\mu_2\lambda_2(\mu_2(\lambda_2-\lambda_1)-\theta^1-2\theta^\infty_3)-\mu_2\lambda_1(\theta^0+\theta^\infty_2-\theta^\infty_3)+\theta^\infty_3(\theta^1+\theta^\infty_3)}{\theta^\infty_2-\theta^\infty_3},$$
$$f_3=\frac{(\lambda_2(\mu_2(\lambda_1-\lambda_2)+\theta^1+\theta^\infty_3+\theta^\infty_4)+(\theta^0+\theta^\infty_2-\theta^\infty_4)\lambda_1)((f_1-1)\mu_2-\mu_1)-f_1\theta^\infty_4(\theta^1+\theta^\infty_4)}{\theta^\infty_2-\theta^\infty_4}.$$

The Hamiltonian is given by

(5.89)
$$\begin{aligned}&t\tilde{H}^{D_5}_{Ss}\begin{pmatrix}\theta^0,\theta^1,\theta^\infty_1, & ;t; & \lambda_1,\mu_1\\ \theta^\infty_2,\theta^\infty_3, & & \lambda_1,\mu_1\end{pmatrix}\\
&=t\tilde{H}_{\mathrm{V}}\begin{pmatrix}-\theta^1-\theta^\infty_2-\theta^\infty_3,\theta^0+\theta^\infty_2-\theta^\infty_3 & ;t;\lambda_1,\mu_1\\ -2\theta^0-\theta^1-\theta^\infty_1-\theta^\infty_2 & \end{pmatrix}+t\tilde{H}_{\mathrm{V}}\begin{pmatrix}\theta^0,\theta^1 & ;t;\lambda_2,\mu_2\\ \theta^\infty_3 & \end{pmatrix}\\
&\quad+2\mu_2\lambda_2(\lambda_1-1)(\mu_1(\lambda_1-1)+\theta^0+\theta^1+\theta^\infty_1+\theta^\infty_3).\end{aligned}$$

The gauge parameters satisfy

(5.90)
$$-\frac{t}{u}\frac{du}{dt}=(\lambda_1-1)(2\mu_2\lambda_2+\mu_1(\lambda_1-1)-\theta^1-\theta^\infty_3-\theta^\infty_4)+t-\theta^\infty_1+\theta^\infty_2,$$
(5.91)
$$-\frac{t}{v}\frac{dv}{dt}=(\lambda_1-1)(\mu_1(\lambda_1-1)+\theta^0+\theta^1+\theta^\infty_1+\theta^\infty_3)+\lambda_2(2\mu_2(\lambda_2-1)-\theta^1-2\theta^\infty_3)+t-\theta^0-\theta^\infty_1+\theta^\infty_3,$$
(5.92)
$$-\frac{t}{w}\frac{dw}{dt}=2(\lambda_1-1)(\mu_1\lambda_1+\mu_2\lambda_2)+\lambda_1(\theta^0+\theta^\infty_1-\theta^\infty_4)+t-2\theta^0-\theta^1-2\theta^\infty_1.$$

When we change the canonical variables as

$$\lambda_1=1-\frac{1}{q_2},\ \mu_1=q_2(q_2p_2+\theta^0+\theta^1+\theta^\infty_1+\theta^\infty_3),\ \lambda_2=1-\frac{1}{q_1},\ \mu_2=q_1(q_1p_1-\theta^1-\theta^\infty_3),$$

then we obtain

(5.93)
$$\begin{aligned}t H^{D_5}_{Ss}\begin{pmatrix}\theta^0,\theta^1,\theta^\infty_1, & ;t; & q_1,p_1\\ \theta^\infty_2,\theta^\infty_3, & & q_2,p_2\end{pmatrix}&=tH_{\mathrm{V}}\begin{pmatrix}\theta^\infty_3,\theta^0-\theta^1 & ;t;q_1,p_1\\ \theta^1 & \end{pmatrix}\\
&\quad+tH_{\mathrm{V}}\begin{pmatrix}-2\theta^0-3\theta^1-\theta^\infty_1-\theta^\infty_2-2\theta^\infty_3,-\theta^0-\theta^1-2\theta^\infty_2 & ;t;q_2,p_2\\ \theta^0+2\theta^1+\theta^\infty_2+\theta^\infty_3 & \end{pmatrix}\\
&\quad+2p_2q_1(p_1(q_1-1)-\theta^1-\theta^\infty_3).\end{aligned}$$

Singularity pattern: 3+1

Spectral type: $((11))((11)),31$

Riemann scheme is given by

$$\left(\begin{array}{c|ccc}x=0 & \multicolumn{3}{c}{\overbrace{\qquad x=\infty\qquad}}\\ 0 & 0 & 0 & \theta^\infty_1\\ 0 & 0 & 0 & \theta^\infty_2\\ 0 & 1 & -t & \theta^\infty_3\\ \theta^0 & 1 & -t & \theta^\infty_4\end{array}\right),$$

and the Fuchs-Hukuhara relation is written as $\theta^0+\theta^\infty_1+\theta^\infty_2+\theta^\infty_3+\theta^\infty_4=0$.

The system of deformation equation is expressed as

$$
\begin{cases}
\dfrac{\partial Y}{\partial x} = \left(A_{\infty}^{(-2)} x + A_{\infty}^{(-1)} + \dfrac{A_0^{(0)}}{x} \right) Y, \\[2ex]
\dfrac{\partial Y}{\partial t} = (E_2 \otimes I_2 x + B_1) Y.
\end{cases}
\tag{5.94}
$$

Here

$$
A_{\xi}^{(-k)} = U^{-1} \hat{A}_{\xi}^{(-k)} U, \quad B_1 = U^{-1} \hat{B}_1 U, \quad U = \mathrm{diag}(1, u, v, w),
$$

$$
\hat{A}_{\infty}^{(-2)} = -E_2 \otimes I_2, \quad E_2 = \mathrm{diag}(0, 1),
$$

$$
\hat{A}_{\infty}^{(-1)} = \begin{pmatrix}
0 & 0 & f_4 \left(f_3 + \theta_3^{\infty} - \frac{\theta_3^{\infty}}{\theta_3^{\infty} - \theta_4^{\infty}} f_1 \right) & -f_4 (f_2 - p_2 q_1 + \theta_4^{\infty}) \\
0 & 0 & -f_4 (f_3 + \theta_3^{\infty}) + \frac{\theta_3^{\infty}}{\theta_3^{\infty} - \theta_4^{\infty}} f_1 (f_4 + 1) & f_4 (f_2 - p_2 q_1 + \theta_4^{\infty}) + \theta_4^{\infty} \\
1 + \frac{1}{f_4} & 1 & t & 0 \\
\frac{f_1 (1 + f_4^{-1})}{\theta_4^{\infty} - \theta_3^{\infty}} + 1 & \frac{f_1}{\theta_4^{\infty} - \theta_3^{\infty}} + 1 & 0 & t
\end{pmatrix},
$$

$$
\hat{A}_0^{(0)} = \begin{pmatrix} -p_2 f_4 \\ -p_1 + p_2 f_4 \\ 1 \\ \frac{f_1}{\theta_4^{\infty} - \theta_3^{\infty}} \end{pmatrix} \begin{pmatrix} q_1 + \frac{q_2}{f_4} & q_1 & f_3 & p_2 q_1 - f_2 \end{pmatrix}, \quad \hat{B}_1 = \begin{pmatrix} O & -[A_{\infty}^{(-1)}]_{1,2} \\ -[A_{\infty}^{(-1)}]_{2,1} & O \end{pmatrix},
$$

$$
\begin{aligned}
&f_1 = p_1 q_1 - \theta_2^{\infty} - \theta_4^{\infty}, \quad f_2 = p_2 q_2 - \theta_1^{\infty} - \theta_4^{\infty}, \\
&f_3 = \frac{(p_1 q_1 - \theta_2^{\infty} - \theta_3^{\infty})(p_2 q_2 - \theta_1^{\infty} - \theta_3^{\infty}) - p_2 q_1 (p_1 q_1 - \theta_2^{\infty} - \theta_4^{\infty})}{\theta_4^{\infty} - \theta_3^{\infty}}, \quad f_4 = \frac{f_1 - p_1 q_2}{\theta_2^{\infty} - \theta_1^{\infty}}.
\end{aligned}
$$

The Hamiltonian is given by

$$
\begin{aligned}
&H_{NY}^{A_4} \left({-\theta_2^{\infty} - \theta_3^{\infty}, \theta_2^{\infty} + \theta_4^{\infty} \atop -\theta_1^{\infty} - \theta_4^{\infty}, \theta_1^{\infty}} ; t; {q_1, p_1 \atop q_2, p_2} \right) \\
&\quad = H_{\mathrm{IV}} \left(\theta_2^{\infty} + \theta_4^{\infty}, -\theta_1^{\infty} - \theta_4^{\infty}; t; q_1, p_1 \right) + H_{\mathrm{IV}} \left(\theta_3^{\infty}, \theta^0; t; q_2, p_2 \right) + 2 q_1 p_1 p_2.
\end{aligned}
\tag{5.95}
$$

The gauge parameters satisfy

$$
\frac{1}{u} \frac{du}{dt} = \frac{(\theta_1^{\infty} - \theta_2^{\infty}) p_1}{p_1 q_2 - f_1}, \quad \frac{1}{v} \frac{dv}{dt} = -q_1 - t + \frac{(\theta_1^{\infty} - \theta_2^{\infty}) p_1}{p_1 q_2 - f_1}, \quad \frac{1}{w} \frac{dw}{dt} = p_1 - t + \frac{(\theta_1^{\infty} - \theta_2^{\infty}) p_1}{p_1 q_2 - f_1}.
\tag{5.96}
$$

Singularity pattern: 2+2

Spectral type: $(2)(2), (111)(1)$

Riemann scheme is given by

$$
\left(\begin{array}{cccc}
\multicolumn{2}{c}{\overbrace{\quad x = 0 \quad}} & \multicolumn{2}{c}{\overbrace{\quad x = \infty \quad}} \\
0 & 0 & t & \theta_1^{\infty} \\
0 & 0 & 0 & \theta_2^{\infty} \\
1 & \theta^0 & 0 & \theta_3^{\infty} \\
1 & \theta^0 & 0 & \theta_4^{\infty}
\end{array} \right),
$$

and the Fuchs-Hukuhara relation is written as $2\theta^0 + \theta_1^{\infty} + \theta_2^{\infty} + \theta_3^{\infty} + \theta_4^{\infty} = 0$.

The system of deformation equations is expressed as

$$
\begin{cases}
\dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0^{(-1)}}{x^2} + \dfrac{A_0^{(0)}}{x} + A_{\infty} \right) Y, \\[2ex]
\dfrac{\partial Y}{\partial t} = (-E_1 x + B) Y.
\end{cases}
\tag{5.97}
$$

Here

$$A_\xi^{(k)} = U^{-1}\hat{A}_\xi{}^{(k)}U, \quad A_\infty = -tE_1, \quad B = U^{-1}\hat{B}U, \quad U = \mathrm{diag}(1,u,v,w), \quad E_1 = \mathrm{diag}(1,0,0,0),$$

$$\hat{A}_0{}^{(0)} = \begin{pmatrix} -\theta_1^\infty & a_{12} & (1-f_1)q_1 - q_2 & -q_1 \\ -p_1q_1 + \theta^0 + \theta_2^\infty + \theta_4^\infty & -\theta_2^\infty & 0 & 0 \\ a_{31} & 0 & -\theta_3^\infty & 0 \\ a_{41} & 0 & 0 & -\theta_4^\infty \end{pmatrix},$$

$$\hat{A}_0{}^{(-1)} = \begin{pmatrix} I_2 \\ B_0^{(-1)} \end{pmatrix} \begin{pmatrix} I_2 - C_0^{(-1)}B_0^{(-1)} & C_0^{(-1)} \end{pmatrix}, \quad B_0^{(-1)} = \begin{pmatrix} p_2 & f_2 \\ p_1 + (1-f_1)p_2 & f_3 \end{pmatrix},$$

$$C_0^{(-1)} = \begin{pmatrix} f_1 & 1 \\ f_1 - 1 & 1 \end{pmatrix}, \quad \hat{B} = \frac{1}{t}\begin{pmatrix} 0 & (A_0^{(-1)})_{12} & (A_0^{(-1)})_{13} & (A_0^{(-1)})_{14} \\ (A_0^{(-1)})_{21} & 0 & 0 & 0 \\ (A_0^{(-1)})_{31} & 0 & 0 & 0 \\ (A_0^{(-1)})_{41} & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
&(\theta_3^\infty - \theta_4^\infty)f_1 = -p_1(q_1 - q_2) + \theta^0 + \theta_2^\infty + \theta_3^\infty, \\
&(\theta_2^\infty - \theta_3^\infty)f_2 = (q_1 - q_2)p_2(1-p_2) - p_2(\theta^0 + \theta_2^\infty + \theta_3^\infty) + \theta_3^\infty, \\
&(\theta_4^\infty - \theta_2^\infty)f_3 = ((1-f_1)p_2 + p_1)((q_1 - q_2)(p_2 - 1) + \theta^0 + \theta_2^\infty + \theta_3^\infty) + f_1\theta_4^\infty, \\
&a_{12} = f_3q_1 + p_2(q_2 - q_1) + f_2((f_1 - 1)q_1 + q_2) - \theta^0 - \theta_2^\infty - \theta_3^\infty, \\
&a_{31} = p_2(\theta_3^\infty - \theta_1^\infty) + p_2q_2(1-p_2) - p_1q_1(f_2 + p_2) - \theta_3^\infty + f_2(\theta^0 + \theta_2^\infty + \theta_4^\infty), \\
&a_{41} = (p_1q_1 + (p_2 - 1)q_2 + \theta_1^\infty - \theta_4^\infty)((f_1 - 1)p_2 - p_1) + f_3(-p_1q_1 + \theta^0 + \theta_2^\infty + \theta_4^\infty) + (f_1 - 1)\theta_4^\infty.
\end{aligned}$$

The Hamiltonian is given by

$$\begin{aligned}
(5.98)\qquad & tH_{Ss}^{D_4}\left(\begin{matrix}\theta^0, \theta_1^\infty, \\ \theta_2^\infty, \theta_3^\infty, \end{matrix}; t; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix}\right) \\
&= tH_{\mathrm{III}}(D_6)\left(\theta^0 + \theta_1^\infty + \theta_3^\infty, -\theta^0 - 2\theta_4^\infty; t; q_1, p_1\right) + tH_{\mathrm{III}}(D_6)\left(-\theta_3^\infty, -\theta^0 - 2\theta_3^\infty; t; q_2, p_2\right) \\
&\quad + 2p_2q_1(p_1q_1 + \theta^0 + \theta_1^\infty + \theta_3^\infty).
\end{aligned}$$

The gauge parameters satisfy

$$(5.99)\qquad \frac{t}{u}\frac{du}{dt} = (1 - 2p_2)q_1 + \theta_1^\infty - \theta_2^\infty, \quad \frac{t}{v}\frac{dv}{dt} = (1 - 2p_2)q_2 + \theta^0 + \theta_1^\infty + \theta_3^\infty,$$

$$(5.100)\qquad \frac{t}{w}\frac{dw}{dt} = (1 - 2p_1 - 2p_2)q_1 - \theta^0 - \theta_2^\infty - \theta_3^\infty.$$

5.4 Matrix Painlevé systems

Singularity pattern: 1+1+1+1

Spectral type: 22,22,22,211

Riemann scheme is given by

$$\begin{pmatrix} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ \theta^0 & \theta^1 & \theta^t & \theta_2^\infty \\ \theta^0 & \theta^1 & \theta^t & \theta_3^\infty \end{pmatrix},$$

and the Fuchs relation is written as $2\theta^0+2\theta^1+2\theta^t+2\theta^\infty_1+\theta^\infty_2+\theta^\infty_3=0$.

The system of deformation equations is expressed as

$$(5.101)\qquad \begin{cases} \dfrac{\partial Y}{\partial x}=\left(\dfrac{A_0}{x}+\dfrac{A_1}{x-1}+\dfrac{A_t}{x-t}\right)Y,\\ \dfrac{\partial Y}{\partial t}=-\dfrac{A_t}{x-t}Y. \end{cases}$$

Here A_0, A_1, and A_t are given as

$$\begin{aligned} A_\xi&=(U\oplus\mathrm{diag}(v,1))^{-1}X^{-1}\hat{A}_\xi X(U\oplus\mathrm{diag}(v,1)),\quad(\xi=0,1,t),\\ \hat{A}_0&=\begin{pmatrix}I_2\\O\end{pmatrix}\left(\theta^0I_2,\frac{1}{t}Q-I_2\right),\quad \hat{A}_1=\begin{pmatrix}I_2\\PQ-\Theta\end{pmatrix}(\theta^1I_2-PQ+\Theta,I_2),\\ \hat{A}_t&=\begin{pmatrix}I_2\\tP\end{pmatrix}\left(\theta^tI_2+QP,-\frac{1}{t}Q\right),\quad U\in GL(2),\quad v\in\mathbb{C}^\times. \end{aligned}$$

Here, Q, P, and Θ are

$$Q=\begin{pmatrix}q_1&1\\-q_2&q_1\end{pmatrix},\quad P=\begin{pmatrix}p_1/2&-p_2\\p_2q_2-\theta-\theta^\infty_1-\theta^\infty_2&p_1/2\end{pmatrix},\quad \Theta=\begin{pmatrix}\theta^\infty_2&\\&\theta^\infty_3\end{pmatrix},$$

where $\theta=\theta^0+\theta^1+\theta^t$.

The matrix X is given by $X=\begin{pmatrix}I_2&O\\Z&I_2\end{pmatrix}$, where

$$Z=(\theta^\infty_1-\Theta)^{-1}[-\theta^1(QP+\theta+\theta^\infty_1)+(QP+\theta+\theta^\infty_1)^2-t(PQ+\theta^t)P].$$

The Hamiltonian is given by

$$\begin{aligned} (5.102)\quad & t(t-1)H^{Mat}_{\mathrm{VI}}\left({-\theta^0-\theta^t-\theta^\infty_1,-\theta^1,\theta^t \atop \theta^0+1,\theta^1+\theta^\infty_3};t;{q_1,p_1\atop q_2,p_2}\right)\\ &=\mathrm{tr}\left[Q(Q-1)(Q-t)P^2+\{(\theta^0+1-(\theta+\theta^\infty_1+\theta^\infty_2)K)Q(Q-1)\right.\\ &\qquad\left.+\theta^t(Q-1)(Q-t)+(\theta+2\theta^\infty_1-1)Q(Q-t)\}P+(\theta+\theta^\infty_1)(\theta^0+\theta^t+\theta^\infty_1)Q\right]. \end{aligned}$$

The gauge parameters satisfy

(5.103)

$$\begin{aligned} t(t-1)\frac{dU}{dt}&=MU,\\ M_{11}&=p_1(2q_1-t)(1-q_1)-(\theta^0+\theta^t+\theta^\infty_1-\theta^\infty_2)q_1+2p_2q_2\\ &\quad+2q_1p_2(q_1(q_1-t-1)+t-q_2)+(\theta^\infty_1-\theta^\infty_3-1)t+\theta^0+\theta^t-\theta^\infty_2+\theta^\infty_3+1,\\ M_{12}&=p_1(2q_1-t)+2p_2q_2+2q_1p_2(t-q_1)+\theta^0+\theta^t+\theta^\infty_1-\theta^\infty_2,\\ M_{21}&=2(\theta+\theta^\infty_1+\theta^\infty_2)q_1(t-q_1)-(2p_2q_2+\theta^0+\theta^t+\theta^\infty_1-\theta^\infty_2)q_2\\ &\quad+p_1q_2(t-2q_1)+2q_1p_2q_2(q_1-t),\\ M_{22}&=((q_1-t)p_1+\theta^0+\theta^t+\theta^\infty_1-\theta^\infty_2)q_1-2tp_2q_2+q_2(4p_2q_1-p_1)+(\theta^0+\theta^1+\theta^\infty_1+\theta^\infty_2)t, \end{aligned}$$

(5.104)

$$\begin{aligned} t(t-1)\frac{1}{v}\frac{dv}{dt}&=2q_1((t+1)p_1+\theta^1+2\theta^\infty_2)-p_1(3q_1^2+t)+2(t+1)p_2q_2+2q_1p_2(q_1(q_1-t-1)+t-3q_2)\\ &\quad+p_1q_2+(\theta^0+\theta^1+2\theta^t+2\theta^\infty_1-1)t+\theta^0+\theta^t-\theta^\infty_2+\theta^\infty_3+1. \end{aligned}$$

Singularity pattern: 2+1+1

Spectral type: (2)(2),22,211

Riemann scheme is given by

$$\begin{pmatrix} x=0 & \overbrace{\quad x=1 \quad} & x=\infty \\ 0 & 0 \quad 0 & \theta^\infty_1 \\ 0 & 0 \quad 0 & \theta^\infty_1 \\ \theta^0 & -t \quad \theta^1 & \theta^\infty_2 \\ \theta^0 & -t \quad \theta^1 & \theta^\infty_3 \end{pmatrix},$$

and the Fuchs-Hukuhara relation is written as $2\theta^0 + 2\theta^1 + 2\theta^\infty_1 + \theta^\infty_2 + \theta^\infty_3 = 0$.

The system of deformation equations is expressed as

$$\left\{ \begin{aligned} \frac{\partial Y}{\partial x} &= \left(\frac{A_0^{(0)}}{x} + \frac{A_1^{(-1)}}{(x-1)^2} + \frac{A_1^{(0)}}{x-1} \right) Y, \\ \frac{\partial Y}{\partial t} &= -\frac{1}{x-1} \left(\frac{A_1^{(-1)}}{t} \right) Y. \end{aligned} \right. \tag{5.105}$$

Here $A_0^{(0)}$, $A_1^{(-1)}$, and $A_1^{(0)}$ are given as follows:

$$\begin{aligned} A_\xi &= (U \oplus \mathrm{diag}(v,1))^{-1} \hat{A}_\xi (U \oplus \mathrm{diag}(v,1)), \\ \hat{A}_1^{(-1)} &= \begin{pmatrix} I_2 \\ -Z+Q \end{pmatrix} (-t(I_2-Q) - tZ, -tI_2) = G_1 \begin{pmatrix} O_2 & O_2 \\ O_2 & -tI_2 \end{pmatrix} G_1^{-1}, \\ \hat{A}_1^{(0)} &= \begin{pmatrix} -(\theta^0+\theta^\infty_1)I_2 + (P+t)Z & P+t \\ \theta^0 Z - Z(P+t)Z & -Z(P+t) - \Theta \end{pmatrix} \\ &= G_1 \begin{pmatrix} O_2 & (P+t)Q - \theta^0 - \theta^1 - \theta^\infty_1 \\ (P+t)Q - P - \theta^0 - \theta^\infty_1 - t & \theta^1 I_2 \end{pmatrix} G_1^{-1}, \\ \hat{A}_0^{(0)} &= \begin{pmatrix} I_2 \\ -Z \end{pmatrix} (\theta^0 I_2 - (P+t)Z, -tI_2 - P), \\ Z &= (\theta^\infty_1 - \Theta)^{-1}[-(P+t)Q(Q-1) + (2\theta^0 + \theta^1 + 2\theta^\infty_1)Q - \theta^0 - \theta^1 - \theta^\infty_1], \\ G_1 &= \begin{pmatrix} I_2 & I_2 \\ Q-Z-I_2 & Q-Z \end{pmatrix}, \quad \left(G_1^{-1} = \begin{pmatrix} Q-Z & -I_2 \\ I_2 - Q + Z & I_2 \end{pmatrix} \right). \end{aligned}$$

Here, Q, P, and Θ are

$$Q = \begin{pmatrix} q_1 & 1 \\ -q_2 & q_1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1/2 & -p_2 \\ p_2q_2 - \theta^0 - \theta^1 - \theta^\infty_1 - \theta^\infty_2 & p_1/2 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta^\infty_2 & \\ & \theta^\infty_3 \end{pmatrix}.$$

The Hamiltonian is given by

$$\begin{aligned} & tH_{\mathrm{V}}^{Mat} \left(\begin{matrix} \theta^\infty_1 - 1, -2\theta^0 - \theta^1 - 2\theta^\infty_1 \\ \theta^0 + \theta^1 + \theta^\infty_1, -\theta^0 - \theta^1 - \theta^\infty_2 - 1 \end{matrix} ; t; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix} \right) \\ &= \mathrm{tr}[P(P+t)Q(Q-1) + (\theta^0 + \theta^1 + \theta^\infty_1)P - (\theta^0 + \theta^1 + 2\theta^\infty_1 - 1)tQ - (2\theta^0 + \theta^1 + 2\theta^\infty_1)PQ]. \end{aligned} \tag{5.106}$$

The gauge parameters satisfy

$$t\frac{dU}{dt} = MU, \tag{5.107}$$

$$M_{11} = (1-2q_1)\left(\frac{3}{2}p_1 + 2t\right) + 2(q_1(q_1-1)p_2 - 2p_2q_2 + \theta^0 + \theta^\infty_1 - \theta^\infty_3),$$

$$M_{12} = (2q_1-1)p_2 - p_1 - 2t,$$

$$M_{21} = (p_2q_2 - \theta^0 - \theta^1 - \theta^\infty_1 - \theta^\infty_2)(1-2q_1) + (p_1+2t)q_2,$$

$$M_{22} = (1-2q_1)\left(\frac{1}{2}p_1 + t\right) - 2p_2q_2 + 4\theta^0 + 3\theta^1 + 4\theta^\infty_1 + 2\theta^\infty_2,$$

$$t\frac{1}{v}\frac{dv}{dt} = p_1(1-2q_1) - 2tq_1 - 2p_2q_2 + 2p_2q_1(q_1-1) + t + 4\theta^0 + 3\theta^1 + 4\theta^\infty_1 + 2\theta^\infty_2. \tag{5.108}$$

Spectral type: (2)(11),22,22

Riemann scheme is given by

$$\left(\begin{array}{ccc} x=0 & x=1 & \overbrace{x=\infty\quad\;}^{} \\ 0 & 0 & 0 \quad \theta^\infty_2 \\ 0 & 0 & 0 \quad \theta^\infty_3 \\ \theta^0 & \theta^1 & t \quad \theta^\infty_1 \\ \theta^0 & \theta^1 & t \quad \theta^\infty_1 \end{array}\right),$$

and the Fuchs-Hukuhara relation is written as $2\theta^0 + 2\theta^1 + 2\theta^\infty_1 + \theta^\infty_2 + \theta^\infty_3 = 0$.

The system of deformation equations is expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(A_\infty + \dfrac{A^{(0)}_0}{x} + \dfrac{A^{(0)}_1}{x-1}\right)Y, \\ \dfrac{\partial Y}{\partial t} = (-E_2 \otimes I_2 x + B_1)Y, \end{cases} \tag{5.109}$$

where

$$A_\xi = (\mathrm{diag}(1,v) \oplus U)^{-1}\hat{A}_\xi(\mathrm{diag}(1,v) \oplus U),$$

$$A_\infty = \begin{pmatrix} O_2 & O_2 \\ O_2 & -tI_2 \end{pmatrix}, \quad \hat{A}^{(0)}_0 = \begin{pmatrix} QP + \theta^0 + \theta^\infty_1 \\ tI_2 \end{pmatrix}\left(I_2 - Q,\ \frac{1}{t}\{(Q-I_2)QP + (\theta^0 + \theta^\infty_1)Q - \theta^\infty_1\}\right),$$

$$\hat{A}^{(0)}_1 = \begin{pmatrix} (QP + \theta^0 + \theta^\infty_1)(Q-I_2) - \Theta \\ tQ \end{pmatrix}\left(I_2,\ \frac{1}{t}\{(QP + \theta^0 + \theta^1 + \theta^\infty_1 + \Theta)Q^{-1} - QP - \theta^0 - \theta^\infty_1\}\right).$$

Furthermore

$$B_1 = (\mathrm{diag}(1,v) \oplus U)^{-1}\begin{pmatrix} O_2 & \frac{[\hat{A}^{(0)}_0 + \hat{A}^{(0)}_1]_{12}}{t} \\ \frac{[\hat{A}^{(0)}_0 + \hat{A}^{(0)}_1]_{21}}{t} & O_2 \end{pmatrix}(\mathrm{diag}(1,v) \oplus U),$$

where $[\hat{A}^{(0)}_0 + \hat{A}^{(0)}_1]_{ij}$ is the (i,j)-block of the matrix $\hat{A}^{(0)}_0 + \hat{A}^{(0)}_1$. Here, Q, P, and Θ are

$$Q = \begin{pmatrix} q_1 & 1 \\ -q_2 & q_1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1/2 & -p_2 \\ p_2q_2 - \theta^0 - \theta^1 - \theta^\infty_1 - \theta^\infty_2 & p_1/2 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta^\infty_2 & \\ & \theta^\infty_3 \end{pmatrix}.$$

The Hamiltonian is given by

$$tH^{Mat}_{\mathrm{V}}\left({-\theta^0 - \theta^1 - \theta^\infty_1, \theta^0 - \theta^1 \atop \theta^1, \theta^\infty_3}; t; {q_1, p_1 \atop q_2, p_2}\right) = \mathrm{tr}[Q(Q-1)P(P+t) + (\theta^0 - \theta^1)QP + \theta^1 P + (\theta^0 + \theta^\infty_1)tQ]. \tag{5.110}$$

The gauge parameters satisfy

$$t\frac{dU}{dt}=\begin{pmatrix} tq_1-\theta^\infty_1+\theta^\infty_2+1 & t \\ -tq_2 & 2p_1q_1+3tq_1-p_1+2p_2q_2-2q_1p_2(q_1-1)-\eta-t \end{pmatrix}U, \tag{5.111}$$

$$t\frac{1}{v}\frac{dv}{dt}=(2q_1-1)p_1-2q_1p_2(q_1-1)+2tq_1+2p_2q_2-t+\theta^0-\theta^1, \quad \eta=\theta^0+3\theta^1+3\theta^\infty_1+\theta^\infty_2-1. \tag{5.112}$$

Singularity pattern: 3+1

Spectral type: ((2))((2)),211

Riemann scheme is given by

$$\left(\begin{array}{ccccc} \multicolumn{3}{c}{\overbrace{\qquad x=0 \qquad}} & & x=\infty \\ 0 & 0 & 0 & & \theta^\infty_1 \\ 0 & 0 & 0 & & \theta^\infty_1 \\ -1 & t & \theta^0 & & \theta^\infty_2 \\ -1 & t & \theta^0 & & \theta^\infty_3 \end{array}\right),$$

and the Fuchs-Hukuhara relation is written as $2\theta^0+2\theta^\infty_1+\theta^\infty_2+\theta^\infty_3=0$.

The system of deformation equations is expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x}=\left(\dfrac{A_0^{(-2)}}{x^3}+\dfrac{A_0^{(-1)}}{x^2}+\dfrac{A_0^{(0)}}{x}\right)Y, \\ \dfrac{\partial Y}{\partial t}=\dfrac{A_0^{(-2)}}{x}Y. \end{cases} \tag{5.113}$$

Here $A_0^{(-2)}$, $A_0^{(-1)}$, and $A_0^{(0)}$ are given as follows:

$$A_\xi^{(k)}=(U\oplus\mathrm{diag}(v,1))^{-1}\hat{A}_\xi^{(k)}(U\oplus\mathrm{diag}(v,1)),$$

$$\hat{A}_0^{(-2)}=\begin{pmatrix} I_2 \\ -Z \end{pmatrix}(-I_2-Z,-I_2), \quad \hat{A}_0^{(-1)}=\begin{pmatrix} PZ+Q+t & P \\ -ZPZ-QZ-ZQ-tZ-Q & -ZP-Q \end{pmatrix},$$

$$\hat{A}_0^{(0)}=-\begin{pmatrix} \theta^\infty_1 I_2 & O \\ O & \Theta \end{pmatrix}, \quad Z=(\theta^\infty_1-\Theta)^{-1}[(P-Q-t)Q-\theta^0-\theta^\infty_1].$$

Here, Q, P, and Θ are

$$Q=\begin{pmatrix} q_1 & 1 \\ -q_2 & q_1 \end{pmatrix}, \quad P=\begin{pmatrix} p_1/2 & -p_2 \\ p_2q_2-\theta^0-\theta^\infty_1-\theta^\infty_2 & p_1/2 \end{pmatrix}, \quad \Theta=\begin{pmatrix} \theta^\infty_2 & \\ & \theta^\infty_3 \end{pmatrix}.$$

The Hamiltonian is given by

$$\begin{aligned} &H^{Mat}_{\mathrm{IV}}\left(\theta^0+2\theta^\infty_1-1,-\theta^0-\theta^\infty_1,\theta^\infty_1-\theta^\infty_2-1;t;\begin{matrix} q_1,p_1 \\ q_2,p_2 \end{matrix}\right) \\ &\quad =\mathrm{tr}[PQ(P-Q-t)-(\theta^0+\theta^\infty_1)P+(\theta^0+2\theta^\infty_1-1)Q]. \end{aligned} \tag{5.114}$$

The gauge parameters satisfy

$$\frac{dU}{dt}=\begin{pmatrix} -\frac{3}{2}p_1+2(p_2+2)q_1+2t & p_2+2 \\ -(p_2+2)q_2+\theta^0+\theta^\infty_1+\theta^\infty_2 & 2q_1+t-\frac{p_1}{2} \end{pmatrix}U, \qquad \frac{1}{v}\frac{dv}{dt}=2(p_2+1)q_1-p_1+t. \tag{5.115}$$

Spectral type: ((2))((11)),22

Riemann scheme is given by

$$\begin{pmatrix} x=0 & \overbrace{\begin{matrix} & x=\infty & \end{matrix}} \\ \begin{matrix} 0 \\ 0 \\ \theta^0 \\ \theta^0 \end{matrix} & \begin{matrix} 0 & 0 & \theta^\infty_2 \\ 0 & 0 & \theta^\infty_3 \\ 1 & -t & \theta^\infty_1 \\ 1 & -t & \theta^\infty_1 \end{matrix} \end{pmatrix},$$

and the Fuchs-Hukuhara relation is written as $2\theta^0 + 2\theta^\infty_1 + \theta^\infty_2 + \theta^\infty_3 = 0$.

The system of deformation equations is expressed as

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(A^{(-2)}_\infty x + A^{(-1)}_\infty + \dfrac{A^{(0)}_0}{x} \right) Y, \\ \dfrac{\partial Y}{\partial t} = (E_2 \otimes I_2 x + B_1) Y, \end{cases} \tag{5.116}$$

where

$$A^{(k)}_\xi = (\mathrm{diag}(1,v) \oplus U)^{-1} \hat{A}^{(k)}_\xi (\mathrm{diag}(1,v) \oplus U),$$
$$\hat{A}^{(-2)}_\infty = \begin{pmatrix} O_2 & O_2 \\ O_2 & -I_2 \end{pmatrix}, \quad \hat{A}^{(-1)}_\infty = \begin{pmatrix} O_2 & PQ-\Theta \\ I_2 & tI_2 \end{pmatrix}, \quad \hat{A}^{(0)}_0 = \begin{pmatrix} -P \\ I_2 \end{pmatrix} (Q,\ QP + \theta^0 I_2),$$
$$B_1 = \begin{pmatrix} O_2 & -[A^{(-1)}_\infty]_{12} \\ -[A^{(-1)}_\infty]_{21} & O_2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}.$$

Here, Q, P, and Θ are

$$Q = \begin{pmatrix} q_1 & 1 \\ -q_2 & q_1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1/2 & -p_2 \\ p_2 q_2 - \theta^0 - \theta^\infty_1 - \theta^\infty_2 & p_1/2 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta^\infty_2 & \\ & \theta^\infty_3 \end{pmatrix}.$$

The Hamiltonian is given by

$$H^{Mat}_{\mathrm{IV}} \left(-\theta^0 - \theta^\infty_1, \theta^0, \theta^\infty_3; t; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix} \right) = \mathrm{tr}[QP(P-Q-t) + \theta^0 P - (\theta^0 + \theta^\infty_1) Q]. \tag{5.117}$$

The gauge parameters satisfy

$$\frac{dU}{dt} = \begin{pmatrix} -q_1 - t & -1 \\ q_2 & p_1 - (2p_2+3)q_1 - 2t \end{pmatrix} U, \quad \frac{1}{v}\frac{dv}{dt} = p_1 - 2(p_2+1)q_1 - t. \tag{5.118}$$

Singularity pattern: 2+2

Spectral type: (2)(2),(2)(11)

Riemann scheme is given by

$$\begin{pmatrix} \overbrace{\begin{matrix} & x=0 & \end{matrix}} & \overbrace{\begin{matrix} & x=\infty & \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ t & \theta^0 \\ t & \theta^0 \end{matrix} & \begin{matrix} 1 & \theta^\infty_1 \\ 1 & \theta^\infty_1 \\ 0 & \theta^\infty_2 \\ 0 & \theta^\infty_3 \end{matrix} \end{pmatrix},$$

and the Fuchs-Hukuhara relation is written as $2\theta^0 + 2\theta^\infty_1 + \theta^\infty_2 + \theta^\infty_3 = 0$.

The system of deformation equations is expressed as

$$\text{(5.119)}\qquad \begin{cases} \dfrac{\partial Y}{\partial x} = \left(\dfrac{A_0^{(-1)}}{x^2} + \dfrac{A_0^{(0)}}{x} + A_\infty \right) Y, \\ \dfrac{\partial Y}{\partial t} = -\dfrac{1}{x} \left(\dfrac{A_0^{(-1)}}{t} \right) Y. \end{cases}$$

Here $A_0^{(-1)}$, $A_0^{(0)}$, and A_∞ are given as follows:

$$\begin{aligned} A_\xi^{(k)} &= (U \oplus \mathrm{diag}(v,1))^{-1} \hat{A}_\xi^{(k)} (U \oplus \mathrm{diag}(v,1)), \\ \hat{A}_0^{(-1)} &= \begin{pmatrix} I_2 \\ P \end{pmatrix} (t(1-P), tI_2), \quad \hat{A}_0^{(0)} = \begin{pmatrix} -\theta_1^\infty I_2 & -Q \\ -Z & -\Theta \end{pmatrix}, \quad \hat{A}_\infty = \begin{pmatrix} -I_2 & O \\ O & O \end{pmatrix}, \\ Z &= (QP + \theta^0 + 2\theta_1^\infty)P - (QP + \theta^0 + \theta_1^\infty). \end{aligned}$$

Here, Q, P, and Θ are

$$Q = \begin{pmatrix} q_1 & 1 \\ -q_2 & q_1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1/2 & -p_2 \\ p_2 q_2 - \theta^0 - \theta_1^\infty - \theta_2^\infty & p_1/2 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta_2^\infty & \\ & \theta_3^\infty \end{pmatrix}.$$

The Hamiltonian is given by

(5.120)

$$tH_{\mathrm{III}(D_6)}^{Mat} \left(\theta^0 + \theta_1^\infty, -\theta^0 - 2\theta_1^\infty, -\theta_2^\infty; t; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix} \right) = \mathrm{tr}[Q^2P^2 - (Q^2 - (\theta^0 + 2\theta_1^\infty)Q - t)P - (\theta^0 + \theta_1^\infty)Q].$$

The gauge parameters satisfy

$$\text{(5.121)}\qquad t\frac{dU}{dt} = \begin{pmatrix} (2p_2q_1 - p_1 + 1)q_1 & p_1 - 2p_2q_1 - 1 \\ 2q_1(p_2q_2 - \theta^0 - \theta_1^\infty - \theta_2^\infty) - (p_1 - 1)q_2 & 2p_2q_2 + (p_1 - 1)q_1 + \theta^0 + 2\theta_1^\infty \end{pmatrix} U,$$

$$\text{(5.122)}\qquad t\frac{1}{v}\frac{dv}{dt} = 2p_2(q_1^2 - q_2) - 2(p_1 - 1)q_1 + \theta^0 + 2\theta_2^\infty.$$

Singularity pattern: 4

Spectral type: (((2)))((((11)))

Riemann scheme is given by

$$\left(\begin{array}{cccc} \multicolumn{4}{c}{\overbrace{\qquad x = \infty \qquad}} \\ 0 & 0 & 0 & \theta_1^\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ -1 & 0 & -t & \theta_2^\infty \\ -1 & 0 & -t & \theta_3^\infty \end{array} \right),$$

and the Fuchs-Hukuhara relation is written as $2\theta_1^\infty + \theta_2^\infty + \theta_3^\infty = 0$.

The system of deformation equations is expressed as

$$\text{(5.123)}\qquad \begin{cases} \dfrac{\partial Y}{\partial x} = \left(A_\infty^{(-3)} x^2 + A_\infty^{(-2)} x + A_\infty^{(-1)} \right) Y, \\ \dfrac{\partial Y}{\partial t} = (A_\infty^{(-3)} x + B_1) Y, \end{cases}$$

where

$$A_\infty^{(k)} = (U \oplus \mathrm{diag}(v,1))^{-1} \hat{A}_\infty^{(k)} (U \oplus \mathrm{diag}(v,1)),$$

$$\hat{A}_\infty^{(-3)} = \begin{pmatrix} O & O \\ O & I_2 \end{pmatrix}, \quad \hat{A}_\infty^{(-2)} = \begin{pmatrix} O & I_2 \\ -P+Q^2+t & O \end{pmatrix}, \quad \hat{A}_\infty^{(-1)} = \begin{pmatrix} -P+Q^2+t & Q \\ (P-Q^2-t)Q-\Theta & P-Q^2 \end{pmatrix},$$

$$B_1 = (U \oplus \mathrm{diag}(v,1))^{-1} \begin{pmatrix} Q & I_2 \\ -P+Q^2+t & O \end{pmatrix} (U \oplus \mathrm{diag}(v,1)).$$

Here Q, P, and Θ are

$$Q = \begin{pmatrix} q_1 & 1 \\ -q_2 & q_1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1/2 & -p_2 \\ p_2q_2 - \theta_1^\infty - \theta_2^\infty & p_1/2 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta_2^\infty & \\ & \theta_3^\infty \end{pmatrix}.$$

The Hamiltonian is given by

$$H_{\mathrm{II}}^{Mat}\left(-\theta_1^\infty+1, \theta_3^\infty+1; t; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix}\right) = \mathrm{tr}[P^2 - (Q^2+t)P + (\theta_1^\infty - 1)Q]. \tag{5.124}$$

The gauge parameters satisfy

$$\frac{dU}{dt} = \begin{pmatrix} 2(q_1+p_2) & 0 \\ 0 & 0 \end{pmatrix} U, \quad \frac{1}{v}\frac{dv}{dt} = 2(q_1+p_2). \tag{5.125}$$

6 Complements to the classification

There is a couple of notices that we should mention concerning 4-dimensional Painlevé-type equations and their degeneration scheme.

First of all, as we have already mentioned, different linear equations sometimes induce the same Painlevé-type equations. In this sense, there is no one-to-one correspondence of linear equations in the scheme to 4-dimensional Painlevé-type equations. However, as far as we know, cases when the same nonlinear equations appear are only those cases when corresponding linear equations transforms one another by the Laplace transformations. It is an interesting question whether or not, in general, linear equations with the same Painlevé-type equation transform one another by the Laplace transformation or some other transformations.

Within this scheme, there are seven deformation equations that have two different linear equations.

Let us see the Laplace transformation in a case when there are one irregular singularity of Poincaré rank 1 and some regular singularities. In this case, the linear equation can be expressed as

$$\frac{d}{dx}Y = \left[Q\left(xI_l - T\right)^{-1} P + S\right] Y, \tag{6.1}$$

where $l = \sum_{i=1}^n \mathrm{rank} A_i$, Q is $m \times l$ matrix, and P is $l \times m$ matrix. Matrices T and S are diagonal.

This equation can be rewritten as

$$\begin{pmatrix} \frac{d}{dx} - S & -Q \\ -P & xI_l - T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = 0. \tag{6.2}$$

Let us apply the Laplace transformation $(x, d/dx) \mapsto (-d/d\xi, \xi)$. Regarding this equation as equation of $\hat{Z}$, the equation reads

$$\frac{d}{d\xi}\hat{Z} = -\left[P\left(\xi I_m - S\right)^{-1} Q + T\right] \hat{Z}, \tag{6.3}$$

which is similar to the original one. Such calculation tells us four correspondences:

$$
\begin{aligned}
&(1)(1),11,11,11 \leftrightarrow (1)(1)(1),21,21 \qquad &&(2)(1),111,111 \leftrightarrow (11)(11),31,21\\
&(2)(2),31,1111 \leftrightarrow (111)(1),22,22 \qquad &&(2)(2),22,211 \leftrightarrow (2)(11),22,22.
\end{aligned}
$$

***Remark* 6.1.** If we put $S = \mathrm{diag}(0,1,t)$, $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the above correspondence, we obtain transformation that leads equation with spectral type $(2)(1),(1)(1)(1)$ to $\frac{3}{2}+1+1+1$-type degenerate Garnier system. $\square$

When the Poincaré rank is 2, calculation becomes more complicated. For an equation

$$
\frac{d}{dx}Y = \left[Q\left(xI_l - T\right)^{-1}P + S_0 + S_1 x\right]Y, \tag{6.4}
$$

let us put $S_1 = \mathrm{diag}(a_1,\ldots,a_k,0,\ldots,0)$, and $\tilde{S_1} = \mathrm{diag}(a_1,\ldots,a_k)$D We also assume that $S_0 = \begin{pmatrix} S_0^{11} & S_0^{12} \\ S_0^{21} & S_0^{22} \end{pmatrix}$, $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, $P = (P_1, P_2)$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$. Here, S_0^{11} is $k \times k$ matrixCQ_1 is $k \times l$, Q_2 is $(m-k)\times l$ matrix, P_1 is $l \times k$, and P_2 is $l \times (m-k)$ matrixD

We can rewrite the equation as

$$
\begin{pmatrix} \frac{d}{dx} - S_0^{11} - \tilde{S}_1 x & -S_0^{12} & -Q_1 \\ -S_0^{21} & \frac{d}{dx} - S_0^{22} & -Q_2 \\ -P_1 & -P_2 & xI_l - T \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Z \end{pmatrix} = 0. \tag{6.5}
$$

If we do the Laplace transformation $(x, d/dx) \mapsto (-d/d\xi, \xi)$, we can eliminate

$$
\hat{Y_2} = (\xi I_{m-k} - S_0^{22})^{-1}(S_0^{21}\hat{Y_1} + Q_2\hat{Z}). \tag{6.6}
$$

Thus, the equation becomes

$$
\begin{aligned}
\frac{d}{d\xi}\begin{pmatrix} \hat{Y_1} \\ \hat{Z} \end{pmatrix}
&= \left[\begin{pmatrix} \tilde{S_1}^{-1}S_0^{12} \\ -P_2 \end{pmatrix}(\xi I_{m-k} - S_0^{22})^{-1}(S_0^{21}, Q_2) + \begin{pmatrix} \tilde{S_1}^{-1}S_0^{11} & \tilde{S_1}^{-1}Q_1 \\ -P_1 & -T \end{pmatrix} - \xi\begin{pmatrix} \tilde{S}_1^{-1} & \\ & O \end{pmatrix}\right]\begin{pmatrix} \hat{Y_1} \\ \hat{Z} \end{pmatrix}.
\end{aligned} \tag{6.7}
$$

Such calculation tells us three correspondences:

$$
\begin{aligned}
&((1))((1)),11,11 \leftrightarrow ((1)(1))((1)),21 \qquad &&((11))((1)),111 \leftrightarrow ((11))((11)),31\\
&((2))((2)),211 \leftrightarrow ((11))((2)),22.
\end{aligned}
$$

***Remark* 6.2.** If we put $T = \mathrm{diag}(0,1)$, $S_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we can obtain transformation that leads degenerate Garnier system of type $\frac{5}{2}+1+1$ to the Painlevé-type equation associated with linear equation of type $(((1)(1)))(((1)))$. $\square$

We should also mention that, sometimes, the degenerate linear equations may admit more freedoms of deformation than the source equation. In our degeneration scheme, such phenomena can be only seen in the degeneration of Fuji-Suzuki system.

What can be counted as freedoms of deformation are positions of singular points and data in Riemann scheme except exponents of formal monodromies. If we exclude parameters that can be normalized to 0 or 1 or other constants by automorphisms of $\mathbb{P}^1$, that is to say, linear fractional transformations and gauge transformations, then the rest is deformation parameters and the number of deformation parameters is dimension of the deformation.

Within the degenerations of Fuji-Suzuki system, four Painlevé-type equations associated with linear equations of spectral types $(1)(1)(1), 21, 21$ and $((1)(1))((1)), 21$ and $(2)(1), (1)(1)(1)$ and $(((1)(1)))(((1)))$ are expressed as partial differential equations with two independent variables. For example, when the spectral type is $(1)(1)(1), 21, 21$, we can assume that the singular points are at $0, 1, \infty$, and that the Riemann scheme can be reduced by gauge transformation of scalar matrices as

$$\left(\begin{array}{ccc} x=0 & x=1 & \overbrace{\begin{array}{cc} \quad & x=\infty \quad \end{array}} \\ 0 & 0 & \begin{array}{cc} 0 & \theta^\infty_1 \end{array} \\ 0 & 0 & \begin{array}{cc} -t_1 & \theta^\infty_2 \end{array} \\ \theta^0 & \theta^1 & \begin{array}{cc} -t_2 & \theta^\infty_3 \end{array} \end{array}\right).$$

In this case, t_1 and t_2 become deformation parameters. Its deformation equation is expressed by $H^{2+1+1+1}_{Gar}$. This is the reason why degenerate Garnier systems appear in the degeneration scheme of Fuji-Suzuki system. On the other hand, we can also consider usual degeneration from $21, 21, 111, 111$:

$$\begin{aligned}
&x \to (1-\varepsilon t x)/(1-\varepsilon t),\ t \to 1/(1-\varepsilon t),\\
&\theta^0_i \to \eta_i \varepsilon^{-1},\ \theta^t \to \theta^0,\ \theta^1 \to \theta^1,\\
&\theta^\infty_1 \to -\eta_1\varepsilon^{-1} + \theta^\infty_1,\ \theta^\infty_2 \to -\eta_2\varepsilon^{-1} + \theta^\infty_2,\ \theta^\infty_3 \to \theta^\infty_3,\\
&q_1 \to \frac{(1-\varepsilon t)q_1}{(1-\varepsilon t)q_1 - 1},\ p_1 \to -(1-\varepsilon t)q_1(p_1 q_1 - \theta^\infty_2)\left(1 - \frac{1}{(1-\varepsilon t)q_1}\right),\\
&q_2 \to \frac{(1-\varepsilon t)q_2}{(1-\varepsilon t)q_2 - 1},\ p_2 \to -(1-\varepsilon t)q_2(p_2 q_2 - \theta^\infty_3)\left(1 - \frac{1}{(1-\varepsilon t)q_2}\right).
\end{aligned}$$

This yields Hamiltonian

$$\begin{aligned}
&tH\left(\begin{matrix}\theta^\infty_1, \theta^\infty_2, \theta^\infty_3 \\ \theta^0, \theta^1\end{matrix}; \eta_1, \eta_2; t; \begin{matrix} q_1, p_1 \\ q_2, p_2 \end{matrix}\right)\\
=&tH_{\mathrm{V}}\left(\theta^0, \theta^\infty_2, \theta^1; (\eta_2-\eta_1)t; q_1, p_1\right) + tH_{\mathrm{V}}\left(\theta^0, \theta^\infty_3, \theta^1; -\eta_1 t; q_2, p_2\right)\\
&+ p_1 p_2 (2q_1 q_2 - q_1 - q_2).
\end{aligned}$$

This Hamiltonian is a restriction of $H^{2+1+1+1}_{Gar}$. Similar stories are true for other three systems, and they correspond to $H^{3+1+1}_{Gar}, H^{\frac{3}{2}+1+1+1}_{Gar}$, and $H^{\frac{5}{2}+1+1}_{Gar}$, respectively.

By the way, we omitted Fuchsian equations with only three singular points for they do not admit deformation. Do not they admit deformation when they are degenerated? The answer is yes; some of Fuchsian equations admit deformation when they are degenerated. However, we can see that all the equations derived from the rest 9 equations by confluences of singular points can be transformed to one of the equations in the scheme by the Laplace transformation. Thus, if we put equations of ramified-type aside, then all the equations are included in this list.

According to Oshima's classification of Fuchsian equations with four accessory parameters, those with only three singular points have following spectral types:

211,1111,1111 221,221,11111 32,11111,11111 222,222,2211 33,2211,111111
44,2222,22211 44,332,11111111 55,3331,22222 66,444,2222211.

If we consider confluences of two regular singular points, then we obtain 17 equations:

(11)(1)(1),1111   (1)(1)(1)(1),211   (2)(2)(1),11111   (11)(11)(1),221
(111)(11),11111   (1)(1)(1)(1)(1),32   (2)(2)(2),2211   (2)(2)(11),222
(111)(111),2211   (11)(11)(1)(1),33   (22)(22),22211   (22)(211),2222   (2)(2)(2)(11),44
(1111)(1111),332   (111)(111)(11),44   (222)(2211),444   (22)(22)(211),66.

Among these 17 equations, the following 6 equations do not admit deformations:

(111)(11),11111   (111)(111),2211   (22)(22),22211   (22)(211),2222
(1111)(1111),332   (222)(2211),444.

If we consider the Laplace transformation of 11 equations which admit deformations, then the leading terms at irregular singularities become scalar matrix, since the original ones have only one regular singularity. Thus, we can eliminate these leading terms at irregular singularities by gauge transformation of scalar matrices. As a result, they become Fuchsian equations, and they are already classified.

For example, we can see a correspondence as below:

$$(11)(1)(1), 1111 \leftrightarrow (111), 111, 21, 21 \sim 21, 21, 111, 111.$$

Similarly, we can find correspondences for the rest 10 equations:

$$\begin{array}{ll}
(1)(1)(1)(1), 211 \leftrightarrow 11, 11, 11, 11, 11 & (2)(2)(1), 11111 \leftrightarrow 31, 22, 22, 1111 \\
(11)(11)(1), 221 \leftrightarrow 21, 21, 111, 111 & (1)(1)(1)(1)(1), 32 \leftrightarrow 11, 11, 11, 11, 11 \\
(2)(2)(2), 2211 \leftrightarrow 22, 22, 22, 211 & (2)(2)(11), 222 \leftrightarrow 22, 22, 22, 211 \\
(11)(11)(1)(1), 33 \leftrightarrow 21, 21, 11, 11 & (2)(2)(2)(11), 44 \leftrightarrow 22, 22, 22, 211 \\
(111)(111)(11), 44 \leftrightarrow 211, 1111, 1111 & (22)(22)(211), 66 \leftrightarrow 222, 222, 2211.
\end{array}$$

For the last two equations, their original deformations are trivially solved, and corresponding Fuchsian equations do not admit deformations. Other equations correspond to one of the four Fuchsian equations corresponding to 4-dimensional Painlevé-type equations.

For 9 equations in Oshima's list, if we consider confluences of three regular singular points to one points, then we obtain 5 equations:

((1)(1))((1)((1)))   ((11))((11))((1))   ((1)(1)(1))((1)(1))   ((2))((2))((11))   ((2)(2))((2)(11)).

They are same as one of the degenerate systems of the four 4-dimensional Painlevé-type equations.

In this paper, we depend on the idea that all Painlevé-type equations of unramified non-Fuchsian systems with 4-dimensional phase space are derived from the four Painlevé-type equations of the Fuchsian equations by degeneration processes. If we degenerate equations after applying middle convolution, do not we come up with new equations that are not derived from the original equations? In fact, when the dimension of phase space is equal to or greater than 6, such a case happens. However, when the dimension of phase space is 2 or 4, it is proved that such a case does not happen [12].

A Data on degenerations

In this appendix, we give explicitly the canonical transformations with ε that link two Hamiltonians in each degeneration. The way of degenerations of Hamiltonians are explained in section 4. It is not necessary to change the data that do not appear in the table below. Note that the terms in Hamiltonian that do not contain the canonical variables p_i and q_i are irrelevent to Hamiltonian system, thus we add or subtract such terms as needed.

Here we omit the following five degenerations

$$\begin{aligned}&21,21,111,111 \to (1)(1)(1),21,21, \quad (11)(1),21,111 \to ((1)(1))((1)),21,\\&(2)(1),111,111 \to (2)(1),(1)(1)(1), \quad (11)(1),(11)(1) \to (((1)(1)))(((1))),\\&((11))((1)),111 \to (((1)(1)))(((1)))\end{aligned}$$

since the number of deformation parameters increases.

A.1 Garnier system

1+1+1+1+1 → 2+1+1+1

$$\begin{aligned}&\theta^{t_1} \to -\varepsilon^{-1},\ \theta^{t_2} \to \theta^t,\ \theta_1^\infty \to \theta_2^\infty + \varepsilon^{-1},\ \theta_2^\infty \to \theta_1^\infty,\\&t_1 \to (\varepsilon t_1)^{-1},\ t_2 \to t_2/t_1,\ H_{t_1} \to -\varepsilon t_1^2 H_{t_1} - \varepsilon t_1 t_2 H_{t_2},\ H_{t_2} \to t_1 H_{t_2},\\&q_1 \to \frac{p_1(1-q_1)+p_2(1-q_2)+\theta^1+\theta^t+\theta_1^\infty}{\varepsilon t_1(q_1-1)(p_1(q_1-1)-\theta^1)},\ p_1 \to \varepsilon t_1(q_1-1)(p_1(q_1-1)-\theta^1),\\&q_2 \to -\frac{t_2(q_2-1)(p_2(q_2-1)-\theta^t)}{t_1(q_1-1)(p_1(q_1-1)-\theta^1)},\ p_2 \to -\frac{t_1(q_1-1)(p_1(q_1-1)-\theta^1)}{t_2(q_2-1)}.\end{aligned}$$

2+1+1+1 → 3+1+1

$$\begin{aligned}&\theta^0 \to \theta_2^\infty + \varepsilon^{-2},\ \theta^t \to \theta^0,\ \theta_1^\infty \to \theta_2^\infty,\ \theta_2^\infty \to \theta_1^\infty - \theta_2^\infty - \varepsilon^{-2},\\&t_1 \to -\varepsilon^{-1}t_1 - \varepsilon^{-2},\ t_2 \to -\varepsilon^{-1}t_2 - \varepsilon^{-2},\ H_{t_1} \to -\varepsilon H_{t_1},\ H_{t_2} \to -\varepsilon H_{t_2},\\&q_1 \to \frac{1}{1-\varepsilon q_1},\ p_1 \to (1-\varepsilon q_1)(\varepsilon^{-1}p_1(1-\varepsilon q_1)+\theta^1),\\&q_2 \to \frac{1}{1-\varepsilon q_2},\ p_2 \to (1-\varepsilon q_2)(\varepsilon^{-1}p_2(1-\varepsilon q_2)+\theta^0).\end{aligned}$$

2+1+1+1 → 2+2+1

$$\begin{aligned}&\theta^0 \to -\varepsilon^{-1},\ \theta^t \to \theta^0 + \varepsilon^{-1},\ t_2 \to \varepsilon t_2,\ H_{t_1} \to H_{t_1},\ H_{t_2} \to \varepsilon^{-1}H_{t_2},\\&q_1 \to \frac{1}{1-q_1},\ p_1 \to (1-q_1)\{(1-q_1)p_1+\theta^1\},\ q_2 \to -\frac{1}{\varepsilon q_2},\ p_2 \to q_2(\varepsilon p_2 q_2 - \varepsilon\theta^0 - 1).\end{aligned}$$

2+2+1 → 3+2

$$\begin{aligned}&\theta^1 \to -\varepsilon^{-2},\ \theta_2^\infty \to \theta_2^\infty + \varepsilon^{-2},\ t_1 \to \varepsilon^{-1}t_2 - \varepsilon^{-2},\ t_2 \to \varepsilon^{-1}t_1,\ H_{t_1} \to \varepsilon H_{t_2},\ H_{t_2} \to \varepsilon H_{t_1},\\&q_1 \to \frac{\varepsilon(p_1q_1 - p_2q_2 - \theta_1^\infty)}{\varepsilon(p_1q_1 - p_2q_2 - \theta_1^\infty)(1+\varepsilon q_2)+q_2},\ p_1 \to \frac{1+\varepsilon q_2}{\varepsilon^2 q_2}\{\varepsilon(p_1q_1 - p_2q_2 - \theta_1^\infty)(1+\varepsilon q_2)+q_2\},\\&q_2 \to -t_1\frac{p_1}{q_2},\ p_2 \to \frac{q_1q_2}{t_1}+1.\end{aligned}$$

2+2+1 → 4+1

$$\begin{aligned}&\theta^0 \to -2\varepsilon^{-3},\ \theta^1 \to \theta^0,\ \theta_2^\infty \to \theta_2^\infty + 2\varepsilon^{-3},\ t_1 \to -\varepsilon^{-2}t_2 + \varepsilon^{-3},\ t_2 \to \varepsilon^{-4}t_1 + \varepsilon^{-6},\\&H_{t_1} \to -\varepsilon^2 H_{t_2},\ H_{t_2} \to \varepsilon^4 H_{t_1},\ q_1 \to \frac{q_2}{q_2+\varepsilon p_1},\ p_1 \to \frac{(q_2+\varepsilon p_1)(p_2(q_2+\varepsilon p_1)-\theta^0)}{\varepsilon p_1},\\&q_2 \to \frac{-p_1(1+\varepsilon q_1)+\varepsilon(p_2q_2+\theta_1^\infty)}{\varepsilon^3 p_1},\ p_2 \to 1-\varepsilon^2 p_1.\end{aligned}$$

3+1+1 → 3+2

$$\theta^0 \to -\varepsilon^{-1},\ \theta^1 \to \theta^0 + \varepsilon^{-1},\ t_1 \to t_2 - \varepsilon t_1,\ H_{t_1} \to -\varepsilon^{-1} H_{t_1},\ H_{t_2} \to \varepsilon^{-1} H_{t_1} + H_{t_2},$$
$$q_1 \to \varepsilon t_1 p_1 + q_2,\ p_1 \to \frac{-q_1}{\varepsilon t_1},\ p_2 \to p_2 + \frac{q_1}{\varepsilon t_1}.$$

3+1+1 → 4+1

$$\theta^1 \to \varepsilon^{-6},\ \theta^\infty_2 \to \theta^\infty_2 - \varepsilon^{-6},\ t_1 \to \varepsilon t_1 - 2\varepsilon^{-3},\ t_2 \to -\varepsilon^{-1} t_2 - \varepsilon^{-3},$$
$$H_{t_1} \to \varepsilon^{-1} H_{t_1},\ H_{t_2} \to -\varepsilon H_{t_2},\ q_1 \to -\varepsilon p_1,\ p_1 \to \varepsilon^{-1} q_1 - \varepsilon^{-3},\ q_2 \to \varepsilon^{-1} q_2,\ p_2 \to \varepsilon p_2.$$

3+2 → 5

$$\theta^0 \to 3\varepsilon^{-4},\ \theta^\infty_2 \to \theta^\infty_2 - 3\varepsilon^{-4},\ t_1 \to \varepsilon^{-3} t_1 + \varepsilon^{-4} t_2 + \varepsilon^{-6},\ t_2 \to t_2 - 3\varepsilon^{-2},$$
$$H_{t_1} \to \varepsilon^3 H_{t_1},\ H_{t_2} \to -\varepsilon^{-1} H_{t_1} + H_{t_2},\ q_1 \to \varepsilon^{-3} q_1 + \varepsilon^{-4},\ p_1 \to \varepsilon^3 p_1 - \varepsilon^2 q_2,\ p_2 \to p_2 - \varepsilon^{-1} q_1 - 2\varepsilon^{-2}.$$

4+1 → 5

$$\theta^0 \to -\varepsilon^{-12},\ \theta^\infty_1 \to \theta^\infty_1,\ t_1 \to -\varepsilon t_1 + \varepsilon^{-2} t_2 + \frac{3}{4}\varepsilon^{-8},\ t_2 \to -\varepsilon^2 t_2 + \frac{3}{2}\varepsilon^{-4},$$
$$H_{t_1} \to -\varepsilon^{-1} H_{t_1},\ H_{t_2} \to -\varepsilon^{-5} H_{t_1} - \varepsilon^{-2} H_{t_2} - \varepsilon^{-2} q_2,\ q_1 \to -\varepsilon^{-1} q_1 - \varepsilon^{-4}/2,\ p_1 \to -\varepsilon^{-2} q_2,$$
$$q_2 \to \varepsilon^2 \frac{q_2 + \varepsilon^3(q_1 q_2 - p_1) - \varepsilon^6 \theta^\infty_1}{1 + \varepsilon^3 q_1 + \varepsilon^6(p_2 - t_2)},\ p_2 \to \varepsilon^{-2}(p_2 - t_2) + \varepsilon^{-5} q_1 + \varepsilon^{-8}.$$

A.2 Fuji-Suzuki system

1+1+1+1→ 2+1+1

$21,21,111,111 \to (2)(1),111,111$

$$\theta^1 \to -\varepsilon^{-1},\ \theta^t \to \theta^1 + \varepsilon^{-1},\ t \to 1 + \varepsilon t,\ H \to \varepsilon^{-1}(H + t^{-1}(p_1 q_1 + p_2 q_2)),$$
$$q_1 \to 1 + \varepsilon t q_1,\ p_1 \to \varepsilon^{-1} t^{-1} p_1,\ q_2 \to 1 + \varepsilon t q_2,\ p_2 \to \varepsilon^{-1} t^{-1} p_2.$$

$21,21,111,111 \to (11)(1),21,111$

$$\theta^0_1 \to \theta^0_2 - \varepsilon^{-1},\ \theta^0_2 \to \theta^0_1,\ \theta^t \to \varepsilon^{-1},\ t \to \varepsilon t,\ H \to \varepsilon^{-1} H,$$
$$q_1 \to 1/q_1,\ p_1 \to -q_1(p_1 q_1 - \theta^0_1 - \theta^\infty_2),\ q_2 \to 1/q_2,\ p_2 \to -q_2(p_2 q_2 - \theta^\infty_3).$$

2+1+1→ 3+1

$(2)(1),111,111 \to ((11))((1)),111$

$$\theta^0_1 \to \theta^0_2 + \varepsilon^{-2},\ \theta^0_2 \to \theta^0_1,\ \theta^1 \to -\varepsilon^{-2}, t \to -\varepsilon^{-1} t + \varepsilon^{-2},\ H \to -\varepsilon H,$$
$$q_1 \to \varepsilon q_1,\ p_1 \to \varepsilon^{-1} p_1,\ q_2 \to \varepsilon q_2,\ p_2 \to \varepsilon^{-1} p_2.$$

$(11)(1),21,111 \to ((11))((1)),111$

$$\theta^1 \to \varepsilon^{-2},\ \theta^0_2 \to \theta^0_2 - \varepsilon^{-2},\ t \to -\varepsilon^{-1} t - \varepsilon^{-2},\ H \to -\varepsilon H,$$
$$q_1 \to \frac{1}{1 - \varepsilon q_2},\ p_1 \to (1 - \varepsilon q_2)\left(\theta^0_1 + \theta^\infty_2 + \varepsilon^{-1} p_2 - p_2 q_2\right),$$
$$q_2 \to \frac{1}{1 - \varepsilon q_1},\ p_2 \to (1 - \varepsilon q_1)\left(\theta^\infty_3 + \varepsilon^{-1} p_1 - p_1 q_1\right).$$

$(1)(1)(1), 21, 21 \to ((1)(1))((1)), 21$

$$\begin{aligned}
&\theta^1 \to \varepsilon^{-2},\ \theta_1^\infty \to \theta_1^\infty - \varepsilon^{-2},\ t_1 \to -\varepsilon^{-1}t_1 - \varepsilon^{-2},\ t_2 \to -\varepsilon^{-1}t_2 - \varepsilon^{-2},\\
&H_{t_1} \to -\varepsilon H_{t_1},\ H_{t_2} \to -\varepsilon H_{t_2},\\
&q_1 \to \frac{1}{1-\varepsilon q_1},\ p_1 \to (1-\varepsilon q_1)\left(\frac{p_1}{\varepsilon}(1-\varepsilon q_1) + \theta_2^\infty\right),\ q_2 \to \frac{1}{1-\varepsilon q_2},\ p_2 \to (1-\varepsilon q_2)\left(\frac{p_2}{\varepsilon}(1-\varepsilon q_2) + \theta_3^\infty\right).
\end{aligned}$$

2+1+1→ 2+2

$(11)(1), 21, 111 \to (11)(1), (11)(1)$

$$\begin{aligned}
&\theta^1 \to \varepsilon^{-1},\ \theta_1^\infty \to \theta_3^\infty - \varepsilon^{-1},\ \theta_3^\infty \to \theta_1^\infty,\ t \to \varepsilon t,\ H \to \varepsilon^{-1}\left(H - \frac{p_1q_1 + p_2q_2}{t}\right),\\
&q_1 \to -\frac{q_1}{\varepsilon t},\ p_1 \to -\varepsilon t p_1,\ q_2 \to -\frac{q_2}{\varepsilon t},\ p_2 \to -\varepsilon t p_2.
\end{aligned}$$

$(1)(1)(1), 21, 21 \to (2)(1), (1)(1)(1)$

$$\begin{aligned}
&\theta^0 \to -\varepsilon^{-1},\ \theta^1 \to \theta^0 + \varepsilon^{-1},\ t_1 \to \varepsilon t_1,\ t_2 \to \varepsilon t_2,\ H_{t_1} \to \varepsilon^{-1}H_{t_1},\ H_{t_2} \to \varepsilon^{-1}H_{t_2},\\
&q_1 \to -\frac{1}{\varepsilon q_1},\ p_1 \to \varepsilon q_1(p_1q_1 - \theta_2^\infty),\ q_2 \to -\frac{1}{\varepsilon q_2},\ p_2 \to \varepsilon q_2(p_2q_2 - \theta_3^\infty).
\end{aligned}$$

2+2→ 4

$(2)(1), (1)(1)(1) \to (((1)(1)))(((1)))$

$$\begin{aligned}
&\theta_1^\infty \to 2\varepsilon^{-2},\ \theta_2^\infty \to -\theta_2^\infty,\ \theta_3^\infty \to -\theta_3^\infty\ t_1 \to -\varepsilon^{-4}t_1 - \varepsilon^{-6},\ t_2 \to -\varepsilon^{-4}t_2 - \varepsilon^{-6},\ H_{t_1} \to -\varepsilon^4 H_{t_1},\ H_{t_2} \to -\varepsilon^4 H_{t_2}\\
&q_1 \to \varepsilon^{-3}(1 + \varepsilon(q_1 - \theta_2^\infty/p_1)),\ p_1 \to \varepsilon^2 p_1,\ q_2 \to \varepsilon^{-3}(1 + \varepsilon(q_2 - \theta_3^\infty/p_2)),\ p_2 \to \varepsilon^2 p_2.
\end{aligned}$$

3+1→ 4

$((1)(1))((1)), 21 \to (((1)(1)))(((1)))$

$$\begin{aligned}
&\theta^0 \to -\varepsilon^{-6},\ \theta_1^\infty \to \theta_1^\infty + \varepsilon^{-6},\ t_1 \to \varepsilon t_1 - 2\varepsilon^{-3},\ t_2 \to \varepsilon t_2 - 2\varepsilon^{-3},\ H_{t_1} \to \varepsilon^{-1}H_{t_1},\ H_{t_2} \to \varepsilon^{-1}H_{t_2},\\
&q_1 \to \varepsilon^{-1}q_1 + \varepsilon^{-3},\ p_1 \to \varepsilon p_1,\ q_2 \to \varepsilon^{-1}q_2 + \varepsilon^{-3},\ p_2 \to \varepsilon p_2.
\end{aligned}$$

A.3 Sasano system

1+1+1+1→2+1+1

$31, 22, 22, 1111 \to (2)(2), 31, 1111$

$$\begin{aligned}
&\theta^1 \to \theta^1 - \varepsilon^{-1},\ \theta^t \to \varepsilon^{-1},\ t \to 1 + \varepsilon t,\ H \to \varepsilon^{-1}(H + t^{-1}(p_1q_1 + p_2q_2)),\\
&q_1 \to 1 + \varepsilon t q_1,\ p_1 \to \varepsilon^{-1}t^{-1}p_1,\ q_2 \to 1 + \varepsilon t q_2,\ p_2 \to \varepsilon^{-1}t^{-1}p_2.
\end{aligned}$$

$31, 22, 22, 1111 \to (11)(11), 31, 22$

$$\begin{aligned}
&\theta^t \to \varepsilon^{-1},\ \theta_1^\infty \to \theta_3^\infty - \varepsilon^{-1},\ \theta_2^\infty \to \theta_4^\infty - \varepsilon^{-1},\ \theta_3^\infty \to \theta_1^\infty,\ \theta_4^\infty \to \theta_2^\infty,\\
&t \to 1/\varepsilon t,\ H \to -\varepsilon t^2 H - \varepsilon t(p_1q_1 + p_2q_2),\\
&q_1 \to 1/\varepsilon t q_2,\ p_1 \to -\varepsilon t q_2(p_2q_2 - \theta^1 - \theta_2^\infty - \theta_4^\infty),\ q_2 \to 1/\varepsilon t q_1,\ p_2 \to -\varepsilon t q_1(p_1q_1 - \theta_1^\infty).
\end{aligned}$$

$31, 22, 22, 1111 \to (111)(1), 22, 22$

$$\begin{aligned}
&\theta^0 \to \varepsilon^{-1},\ \theta^t \to \theta^0,\ \theta_1^\infty \to \theta_1^\infty - \varepsilon^{-1},\ \ t \to \frac{1}{1-\varepsilon t},\ H \to \varepsilon^{-1}H,\\
&q_1 \to \frac{q_1 - 1}{q_1(1-\varepsilon t)},\ p_1 \to q_1(1-\varepsilon t)(p_1q_1 + \theta^0 + \theta^1 + \theta_1^\infty + \theta_3^\infty),\\
&q_2 \to \frac{q_2 - 1}{q_2(1-\varepsilon t)},\ p_2 \to q_2(1-\varepsilon t)(p_2q_2 - \theta^1 - \theta_3^\infty).
\end{aligned}$$

2+1+1→ 3+1

$(11)(11), 31, 22 \to ((11))((11)), 31$

$$\begin{aligned}
&\theta^1 \to \varepsilon^{-2},\ \theta_3^\infty \to \theta_3^\infty - \varepsilon^{-2},\ \theta_4^\infty \to \theta_4^\infty - \varepsilon^{-2},\ t \to -\varepsilon^{-1}t - \varepsilon^{-2},\ H \to -\varepsilon H,\\
&q_1 \to \frac{1}{1-\varepsilon q_2},\ p_1 \to (1-\varepsilon q_2)(\varepsilon^{-1}p_2(1-\varepsilon q_2) + \theta_1^\infty),\\
&q_2 \to \frac{1}{1-\varepsilon q_1},\ p_2 \to (1-\varepsilon q_1)(\varepsilon^{-1}p_1(1-\varepsilon q_1) + \theta_2^\infty + \theta_4^\infty).
\end{aligned}$$

$(2)(2), 31, 1111 \to ((11))((11)), 31$

$$\begin{aligned}
&\theta^1 \to -\varepsilon^{-2},\ \theta_1^\infty \to \theta_1^\infty + \varepsilon^{-2},\ \theta_2^\infty \to \theta_2^\infty + \varepsilon^{-2},\ t \to -\varepsilon^{-1}t - \varepsilon^{-2},\ H \to -\varepsilon H,\\
&q_1 \to \frac{1}{1-\varepsilon q_1},\ p_1 \to (1-\varepsilon q_1)(\varepsilon^{-1}p_1 - p_1q_1 - \theta^0 - \theta_1^\infty - \theta_3^\infty),\\
&q_2 \to \frac{1}{1-\varepsilon q_2},\ p_2 \to (1-\varepsilon q_2)(\varepsilon^{-1}p_2 - p_2q_2 + \theta_3^\infty).
\end{aligned}$$

2+1+1→ 2+2

$(2)(2), 31, 1111 \to (2)(2), (111)(1)$

$$\begin{aligned}
&\theta^0 \to \varepsilon^{-1},\ \theta^1 \to \theta^0,\ \theta_1^\infty \to \theta_1^\infty - \varepsilon^{-1},\ t \to -\varepsilon t,\ H \to -\varepsilon^{-1}\left(H - \frac{p_1q_1 + p_2q_2}{t}\right),\\
&q_1 \to \varepsilon^{-1}t^{-1}q_1,\ p_1 \to \varepsilon t p_1,\ q_2 \to \varepsilon^{-1}t^{-1}q_2,\ p_2 \to \varepsilon t p_2.
\end{aligned}$$

$(111)(1), 22, 22 \to (2)(2), (111)(1)$

$$\begin{aligned}
&\theta^0 \to \theta^0 - \varepsilon^{-1},\ \theta^1 \to \varepsilon^{-1},\ t \to \varepsilon t,\ H \to \varepsilon^{-1}H,\\
&q_1 \to -\frac{1}{\varepsilon q_1},\ p_1 \to \varepsilon q_1(p_1q_1 + \theta^0 + \theta_1^\infty + \theta_3^\infty),\ q_2 \to -\frac{1}{\varepsilon q_2},\ p_2 \to q_2(\varepsilon(p_2q_2 - \theta_3^\infty) - 1).
\end{aligned}$$

A.4 matrix Painlevé system

1+1+1+1→2+1+1

$22, 22, 22, 211 \to (2)(2), 22, 211$

$$\begin{aligned}
&\theta^1 \to \varepsilon^{-1},\ \theta^t \to \theta^1 - \varepsilon^{-1},\ t \to 1 + \varepsilon t,\ H \to \varepsilon^{-1}H,\\
&Q \to 1 - \varepsilon P,\ P \to \varepsilon^{-1}Q.
\end{aligned}$$

$22, 22, 22, 211 \to (2)(11), 22, 22$

$$\begin{aligned}
&\theta^t \to \varepsilon^{-1},\ \theta_1^\infty \to \theta_1^\infty - \varepsilon^{-1},\ t \to (\varepsilon t)^{-1},\ H \to -\varepsilon t^2 H - \varepsilon t\ \mathrm{tr}(PQ),\\
&Q \to (\varepsilon t)^{-1}Q^{-1},\ P \to -\varepsilon t(QP + \theta^0 + \theta_1^\infty)Q.
\end{aligned}$$

2+1+1→ 3+1

$(2)(2), 22, 211 \to ((2))((2)), 211$

$$\begin{aligned}
&\theta^0 \to \theta^0 - \varepsilon^{-2},\ \theta^1 \to \varepsilon^{-2},\ t \to \varepsilon^{-1}(-t + \varepsilon^{-1}),\ H \to -\varepsilon H,\\
&Q \to \varepsilon Q,\ P \to \varepsilon^{-1}P.
\end{aligned}$$

$(2)(2), 22, 211 \to ((2))((11)), 22$

$$\begin{aligned}
&\theta^1 \to \varepsilon^{-2},\ \theta_2^\infty \to \theta_2^\infty - \varepsilon^{-2},\ \theta_3^\infty \to \theta_3^\infty - \varepsilon^{-2},\ t \to \varepsilon^{-1}(-t - \varepsilon^{-1}),\ H \to -\varepsilon(H + \mathrm{tr}P),\\
&Q \to (1 + \varepsilon P)^{-1},\ P \to \{(P + \varepsilon^{-1})(-P + Q + t) + \theta^0 + 2\theta_1^\infty + \varepsilon^{-2} - 1\}(\varepsilon P + 1).
\end{aligned}$$

$(2)(11), 22, 22 \to ((2))((11)), 22$

$$\begin{aligned}
&\theta^1 \to \varepsilon^{-2},\ \theta^\infty_1 \to \theta^\infty_1 - \varepsilon^{-2},\ t \to \varepsilon^{-1}(-t - \varepsilon^{-1}),\ H \to -\varepsilon H,\\
&Q \to (1 - \varepsilon Q)^{-1},\ P \to \{(Q - \varepsilon^{-1})P + \theta^0 + \theta^\infty_1 - \varepsilon^{-2}\}(\varepsilon Q - 1).
\end{aligned}$$

2+1+1→2+2

$(2)(2), 22, 211 \to (2)(2), (2)(11)$

$$\theta^0 \to \varepsilon^{-1},\ \theta^1 \to \theta^0,\ \theta^\infty_1 \to \theta^\infty_1 - \varepsilon^{-1},\ t \to \varepsilon t,\ H \to \varepsilon^{-1}H,\ Q \to P,\ P \to -Q.$$

$(2)(11), 22, 22 \to (2)(2), (2)(11)$

$$\begin{aligned}
&\theta^0 \to -\varepsilon^{-1},\ \theta^1 \to \theta^0 + \varepsilon^{-1},\ t \to \varepsilon t,\ H \to \varepsilon^{-1}H,\\
&Q \to (-\varepsilon Q)^{-1},\ P \to \varepsilon(QP - \varepsilon^{-1} + \theta^\infty_1)Q.
\end{aligned}$$

2+2→ 4

$(2)(2), (2)(11) \to (((2)))(((11)))$

$$\begin{aligned}
&\theta^0 \to -2\varepsilon^{-3},\ \theta^\infty_1 \to \theta^\infty_1 + 2\varepsilon^{-3},\ t \to -\varepsilon^{-4}t - \varepsilon^{-6},\ H \to -\varepsilon^4(H + \mathrm{tr}Q),\\
&Q \to \varepsilon^{-3}(1 - \varepsilon Q),\ P \to \varepsilon^2(-P + Q^2 + t).
\end{aligned}$$

3+1→ 4

$((2))((2)), 211 \to (((2)))(((11)))$

$$\begin{aligned}
&\theta^0 \to -\varepsilon^{-6},\ \theta^\infty_2 \to \theta^\infty_2 + \varepsilon^{-6},\ \theta^\infty_3 \to \theta^\infty_3 + \varepsilon^{-6},\ t \to \varepsilon t - 2\varepsilon^{-3},\ H \to \varepsilon^{-1}H,\\
&Q \to \varepsilon^{-1}Q + \varepsilon^{-3},\ P \to \varepsilon P + \varepsilon^3(\theta^\infty_1 - \varepsilon^{-6})(\varepsilon^2 Q + 1)^{-1}.
\end{aligned}$$

$((2))((11)), 22 \to (((2)))(((11)))$

$$\begin{aligned}
&\theta^0 \to -\varepsilon^{-6},\ \theta^\infty_1 \to \theta^\infty_1 + \varepsilon^{-6},\ t \to \varepsilon t - 2\varepsilon^{-3},\ H \to \varepsilon^{-1}(H + \mathrm{tr}Q),\\
&Q \to -\varepsilon^{-1}Q + \varepsilon^{-3},\ P \to -\varepsilon(P - Q^2 - t).
\end{aligned}$$

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